A few problems on the Steiner distance and crossing number of graphs

by

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Bachelor of Arts
Minnesota State University-Moorhead 2015

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2019

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Dedication

To Earl, Al, Ron, and Bev for instilling a value of education lasting generations.

Acknowledgments

Were I to thank everyone who contributed to my writing of this dissertation, the length of the document would more than double, a result to which I am not opposed.

First and foremost, I would like to thank my advisors, Dr. Éva Czabarka and Dr. László Székely. Were it not for their guidance, this document, and its many forebearers, would not have been produced in one hundred years, let alone in four. I was truly fortunate to be advised by both of you and have learned much in terms of mathematics as well as navigating teaching, research, the academic job market, and the occasional life lesson.

Thank you to Dr. George Mcnulty and Dr. Edsel Pena for not only serving on my committee, but also teaching me the inner workings of Universal Algebra and Probability theory, respectively. The methods discussed in those classes aided in the completion of this dissertation both directly and indirectly. Additional thanks are due to Dr. Peter Dankelmann for providing crucial encouragement and helpful problems at a time when I doubted my abilities to complete this dissertation and to Dr. Ortrud Oellermann for providing a document which provided the background necessary to begin Chapter 3 of this dissertation.

Thanks must be extended to my many friends who have encouraged me through this process. I would like to thank the members of the Hungarian and Discrete Optimization seminars. Most of you were with me through this dissertation from its inception and provided much encouragement. In particular, I would like to thank Inne Singgih and Zhiyu Wang for the many study sessions, study breaks, and meals we have shared. Additional thanks are due to my friends from Midtown Fellowship and

King's Church for keeping my perspective on things beyond my immediate graduate school stresses.

I would like to extend thanks to my family: to my parents Kim and Cheri for teaching me my first mathematical statements and always supporting my endeavors; to my family-in-law Zeno, Robin, and Bobby for making Columbia home these past four years; and to my wife Berkeley. You were not the first BAR to put a husband through school, and I am continually grateful for your patience, encouragement, and future-driven attitude.

Lastly, I am compelled to recognize that whichever logical statements succeed these comments are due entirely to the mind of our creator and savior Jesus Christ. No written work has directed both my theology and my career more than Job 39.

Soli Deo gloria

The work of this dissertation was supported in part by the grant NSF DMS 1600811.

Abstract

We provide a brief overview of the Steiner ratio problem in its original Euclidean context and briefly discuss the problem in other metric spaces. We then review literature in Steiner distance problems in general graphs as well as in trees.

Given a connected graph G we examine the relationship between the Steiner k-diameter, $\mathrm{sdiam}_k(G)$, and the Steiner k-radius, $\mathrm{srad}_k(G)$. In 1990, Henning, Oellermann and Swart [*Ars Combinatoria* **12** 13-19, (1990)] showed that for any connected graph G, $\mathrm{sdiam}_3(G) \leq \frac{8}{5}\,\mathrm{srad}_3(G)$ and conjectured that for all $k \geq 2$ and a connected graph G, $\mathrm{sdiam}_k(G) \leq \frac{2(k+1)}{2k-1}\,\mathrm{srad}_k(G)$. The paper also included an incorrect proof that $\mathrm{sdiam}_4(G) \leq \frac{10}{7}\,\mathrm{srad}_4(G)$. We provide a correct proof that $\mathrm{sdiam}_4(G) \leq \frac{10}{7}\,\mathrm{srad}_4(G)$ and show that for $k \geq 5$, $\mathrm{sdiam}_k(G) \leq \frac{k+3}{k+1}\,\mathrm{srad}_k(G)$. By construction, we also show that the latter of these bounds is tight for each $k \geq 5$.

We then examine the Steiner distance of large sets in hypercubes. In particular, we show that for $k = O(2^n/n)$, the Steiner k-diameter of the n-cube is $k + \Theta(\frac{2^n}{\sqrt{n}})$ using a recent result of Griggs. This section is a joint work with Éva Czabarka and László Székely.

Finally, we move to structural properties of graphs in the context of crossing numbers. For positive integers n and e, let $\kappa(n, e)$ be the minimum number of crossings among all graphs with n vertices and at least e edges. Under the condition that $n \ll e \ll n^2$, Pach, Spencer, and Tóth [*Discrete and Computational Geometry* **24** 623-644, (2000)] showed that $\kappa(n, e)\frac{n^2}{e^3}$ tends towards a positive constant (called the midrange crossing constant) as $n \to \infty$. We extend their proof to show that the midrange crossing constant exists for graph classes that satisfy a certain set of graph

properties. As a corollary, we show that the the midrange crossing constant exists for the family of bipartite graphs. This section is a joint work with Éva Czabarka, László Székely, and Zhiyu Wang.

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Chapter 1

Prelude: The Euclidean Steiner Tree Problem

1.1 An introduction to the Euclidean Steiner tree problem

The Euclidean Steiner tree problem has a rich history, intermittently examined by mathematicians over several centuries. In their history of the problem, Brazil, Graham, Thomas, and Zachariasen provide a clear formulation of the problem.

Problem 1.1. [9] Given a set N of n points in the plane (often called terminal points), find a system of line segments such that the union of the line segments forms a connected set containing N, and such that the total Euclidean length of the line segments is minimized.

Necessarily, the resulting system of line segments must form a tree, i.e., the resulting structure is connected and contain no cycles. A *minimal spanning tree* of N, $MST(N)$, is a tree having *only* elements of N as vertices and smallest possible length. A naive approach to the Euclidean Steiner tree problem would be to apply efficient algorithms, such as Dijkstra's algorithm (See [17]), to find a minimum spanning tree of N [28]. However, by adding additional points, called Steiner points, it is possible to find trees of potentially smaller length. For example, if we consider the 4 points at the corner of a unit square, one can find a smaller tree spanning N by including a Steiner point at the center of the square. Still more, by including two Steiner points, one can find a still smaller tree spanning N. Figure 1.1 illustrates this fact. The smallest tree spanning N without restricting the number of Steiner points is called a *Steiner tree* for N, and denoted by $ST(N)$.

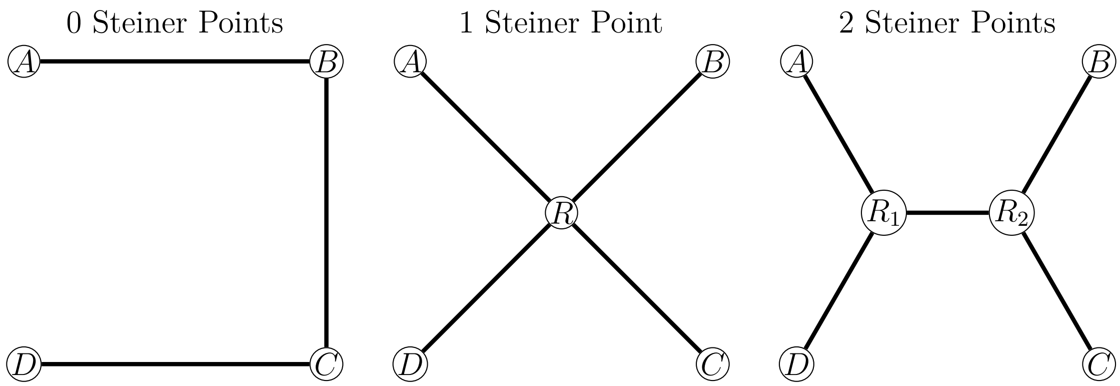

Figure 1.1 Spanning trees for the unit square using 0, 1, or 2 Steiner points. The Steiner points are labeled by the letter R.

1.2 History of the Euclidean Steiner tree problem

Following most named results in mathematics, the origins of the problem are not found with Steiner. The following restriction of the problem was examined by Fermat around 1641.

Problem 1.2 (as translated in [9])**.** [36] Given three points, a fourth is to be found, from which if three straight lines are drawn to the given points, the sum of the three lengths is minimum.

This is a restriction of the Steiner tree problem to three terminal points and a single Steiner point. A solution to the problem was produced by Italian mathematician Evangelista Torricelli in the same century [9]. This restriction of the Steiner tree problem is known as the Fermat–Torricelli problem.

The first known (full) formulation of the Euclidean Steiner tree problem was made in the 1800's by French Mathematician Joseph Diaz Gergonne. In the inaugural volume of his journal *Annales de mathématiques pures et appliquées*, more commonly known as *Annales de Gergonne*, Gergonne introduced the problem gradually through a series of related questions.

Initially, Gergonne proposed what is essentially a retelling of the Fermat–Torricelli problem in a concrete setting.

Problem 1.3 (as translated in [9]). [27] An engineer wishes to establish a communication between three cities, not located in a straight line, by means of a network composed of three branches, leading at one end to the three cities, and meeting at the other end at a single point between these three cities. The question is, how can one locate the point of intersection of the three branches of the network, so that their total length is as small as possible?

A footnote restated the problem in an abstract setting.

Problem 1.4 (as translated in [9]). [27] One can generalize this problem by asking how to determine, on a plane, a point whose sum of distances to a number of arbitrary points located in this plane is minimal. One can even extend to points located in any manner in space.

On a later page of the same volume, the first full version of the Euclidean Steiner tree problem was formulated in the following setting.

Problem 1.5 (As translated in [9]). [27] A number of cities are located at known locations on a plane; the problem is to link them together by a system of canals whose total length is as small as possible.

The problem was later generalized on page 375 of [27]. The author of the article, listed as "Subscriber" and believed in [9] to be Gergonne, elaborates on eleven problems related to the Euclidean Steiner tree problem. In particular, the final of these problems is translated by [9] as follows:

Problem 1.6 (As translated in [9]). [27] Connect any number of given points by a system of lines whose total length is as small as possible.

This is the Euclidean Steiner problem in full generality. Gergonne went on to give a detailed analysis of the problem. A condensed version of Gergonne's treatment of the problem was later published in England by "Gallicus," a Pseudonym for an unknown mathematician. See [9] for an analysis of the Gergonne's treatment of the Euclidean Steiner tree problem as well as the written work of Gallicus.

Following these initial treatments of the Euclidean Steiner tree problem, the problem was discussed by Carl Fredrick Gauss and Christian Schumacher during a correspondence during the 1830s. In their correspondence, Gauss examined the Steiner problem restricted to 4 terminal points [9].

This restriction of the Steiner tree problem was the subject of Karl Bopp's Ph. D. dissertation in 1879 [8]. In his dissertation, Bopp proved several results which were later rediscovered in modern treatments (see [52] and [19]) of the Euclidean Steiner tree problem [9]. The same subject was examined by Hoffman in 1890 [33]. Both Bopp and Hoffman were motivated by Gauss's letter and provided citations for it [9]. Hoffman's paper is of particular interest in that it provides a short discussion of the general Steiner problem with n terminal points.

Following Hoffman's paper, there was little development in the Steiner tree problem until 1934. In this year Jarník and Kössler examined not only the fully generalized Euclidean Steiner tree problem but also extended the problem to higher dimensional Euclidean spaces [36] (translated in [40]). According to [9], the paper was largely ignored by the mathematics community as it was written in Czech. Of particular importance in this paper are proofs that every Steiner point has degree at least 3 and that an angle inside of a Steiner tree has measure at least 120°.

We conclude our historical overview of the Euclidean Steiner tree problem by mentioning two expositions which brought the problem to new heights in the 20th century. In 1941 Richard Courant and Herbert Robbins published their classic book *What is Mathematics?* [13]. Written towards an audience of non-mathematicians, the text bears witness to the simplicity of the Steiner tree problem. It is in this publication that the problem is attributed to Steiner, though no justification is given for the naming convention. Following the publication of the book, the problem gained traction in the the mathematical community. In 1968, Gilbert and Pollack [28] published a comprehensive overview of the research for the Euclidean Steiner tree problem for a professional audience and provided a more robust geometric framework for the problem.

1.3 The general Steiner tree problem and Steiner ratio problem

The Steiner tree problem can be extended to metric spaces beyond the Euclidean plane. A metric space is a set X with a distance function d, mapping X^2 to the non-negative real numbers, satisfying a finite set of axioms. See [20] for a complete treatment of metric spaces. Indeed, Steiner tree problems have been examined with respect to weighted graphs [10], the rectangular metric space [26], as well as general metric spaces [54]. Numerous extensions of the Steiner tree problem have been examined (See [34]), but we limit ourselves to the Steiner tree problem in this exposition.

The difficulty in finding effective solutions to the Steiner tree problem extends to its complexity. Both the Euclidean and rectangular Steiner tree problem have been shown to be NP-hard [26]. In fact, a version of the Steiner tree problem in graphs was included in Karp's original list of NP-complete problems [39].

Given the computational difficulty of finding a Steiner tree for a set of points, significant effort has been put into finding good approximations of Steiner trees (See [54] and [10]). Recall that a minimal spanning tree of a set N is a smallest tree connecting N containing *no* terminal points other than N, i.e. the smallest tree containing N and no Steiner points. Let $d(MST(N))$ denote the weight of a minimal spanning tee of N. Then, if $d(ST(N))$ denotes the weight of a Steiner tree of N, one can analyze the parameter

$$\rho_{(X,d)} := \max_{N \subseteq X} \frac{d(MST(N))}{d(ST(N))}.$$

Here, $\rho_{(X,d)}$ gives the maximum ratio of the weight of a minimal spanning tree of N to the weight of a Steiner tree of N over all $N \subseteq X$. This ratio is known as the *Steiner ratio problem*. For any metric space (X, d), it is known that $\rho_{(X,d)} \leq 2$ and that this bound is tight for certain graphs [58].

Perhaps the most popular version of the Steiner ratio problem is the Euclidean Steiner ratio problem which asks for the value of ρ in the context of the plane with the Euclidean metric. As of yet, the ratio is unknown. However a longstanding conjecture of Gilbert and Pollock seems to be reasonable to this author.

Conjecture 1.7. [28] Suppose that N is a finite set of point in the plane. Then,

$$\max_{N \subseteq X} \frac{d(MST(N))}{d(ST(N))} = \frac{2}{\sqrt{3}}.$$

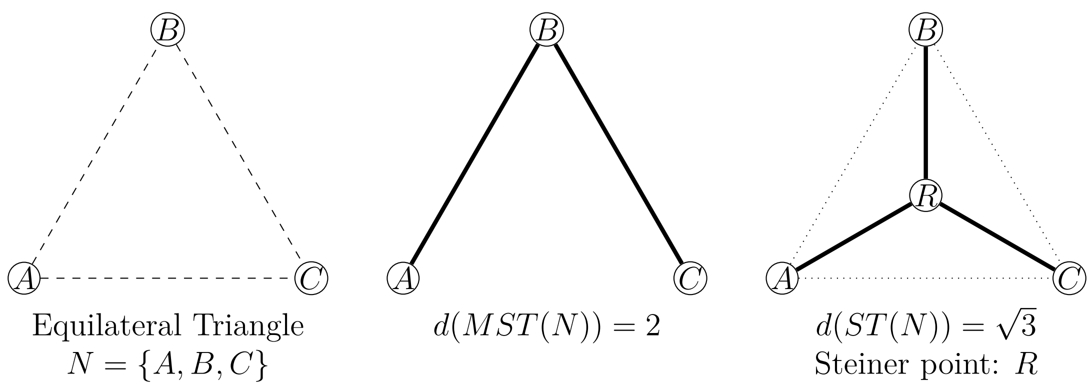

Figure 1.2 An equilateral triangle with unit length. The Steiner point is labeled R.

Should this bound be correct, it is sharp. This can be easily seen by three points in the plane forming an equilateral triangle with sides of unit length as illustrated in Figure 1.2. To create the minimal spanning tree of N, we need two edges of length

2. To create the Steiner tree of N, we add a single Steiner point at the center of the triangle.

A purported proof of Conjecture 1.7 was published in 1992 [18]. This proof, however, was shown to be incorrect by Ivanov and Tuzhilin ten years later [35]. While supplying proof of Conjecture 1.7 would indeed be impressive, we leave that challenge to another day. In fact, the difficulty in finding Steiner trees in a metric space will be avoided entirely. Instead we will restrict ourselves to graphs and (mostly) avoid finding Steiner trees.

Chapter 2

The Steiner Distance in Graphs

2.1 Definitions and examples

Suppose that $G = (V, E)$ is a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We let $|G| = |V(G)|$ denote the order of G and $\|G\| = |E(G)|$ denote the size of G. Given a vertex $v \in V(G)$, the degree of v, $\deg(v)$, is the number of edges incident to that vertex. We let δ denote the minimum degree among all vertices in V. The open neighborhood of v in G is denoted by $N_G(v)$ and is defined as the set of all vertices in G adjacent to v. The closed neighborhood of v in G is denoted by $N_G[v]$ and is defined as the union $N_G(V) \cup \{v\}$.

The distance in G between two vertices $u, v \in V$, denoted $d_G(u, v)$, is the length of the shortest path in G between u and v. When the context is clear, we omit G and write $d(u, v)$ for the distance between u and v in G. If there is no path between u and v in G, we say that $d_G(u, v) = \infty$. The *eccentricity* of a vertex v in G is defined as $\mathrm{e}(v) := \max\{d_G(u, v) : u \in V(G)\}$. The *radius*, $\mathrm{rad}(G)$, is defined as

$$\mathrm{rad}(G) := \min\{e(v) : v \in V(G)\}$$

and the *diameter*, $\mathrm{diam}(G)$, is defined as

$$\mathrm{diam}(G) := \max\{\mathrm{e}(v) : v \in V(G)\}.$$

The center of G, denoted $C(G)$, is the subgraph induced by all vertices $v \in V(G)$ such that $e(v) = \mathrm{rad}(G)$. If H is a subgraph of G and $v \in V(G)$, then *the distance from v to H*, denoted $d_G(v, H)$ is defined as $\min\{d_G(v, u) : u \in V(H)\}$.

The distance between two vertices v and u can be viewed as the minimal size of a connected subgraph (in this case, a path) of G containing v and u. This suggests a generalization of distance. Introduced in [12], the *Steiner distance* in G of a non-empty set $S \subset V(G)$, denoted $d_G(S)$, is defined as the size of the smallest connected subgraph of G containing all elements of S. When the context is clear, we simply write $d_G(S)$ as $d(S)$. In their paper introducing the Steiner distance, Chartrand, Oellermann, and Swart made the following observations.

Observation 2.1. [12] Suppose that H is a connected subgraph of G containing S with $\|H\| = d(S)$. Then,

1. The subgraph H is a tree. Such a tree is called a *Steiner tree* of S.

2. The set of end vertices (vertices of degree 1) of H must be a subset of S.

Consider the graph G illustrated in Figure 2.1. The edges of a Steiner tree for the set $\{A, B, D\}$ are in bold. This Steiner tree contains 5 edges which implies $d(\{A, B, D\}) = 5$. This example illustrates that a Steiner tree is not unique as the shortest path from A to B through D also contains 5 edges.

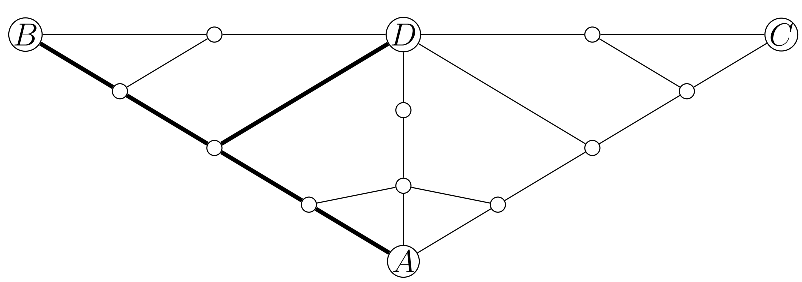

Figure 2.1 The graph G. A Steiner tree of $\{A, B, D\}$ is in bold.

Given an integer $k \geq 2$, the *Steiner k-eccentricity* of a vertex v in G, denoted $\mathrm{e}_k(v)$, is defined as the maximum Steiner distance of all vertex subsets of G of size k

containing v. More succinctly,

$$\mathrm{e}_k(v) = \max_{S \subset V(G), |S|=k} \{d(S) : v \in S\}.$$

The *Steiner k-radius*, denoted $\mathrm{srad}_k(G)$, is then defined as

$$\mathrm{srad}_k(G) := \min\{e_k(v) : v \in G\},$$

while the *Steiner k-diameter*, denoted $\mathrm{sdiam}_k(G)$ is then defined as

$$\mathrm{sdiam}_k(G) := \max\{e_k(v) : v \in G\}.$$

The Steiner k-center, $C_k(G)$, is the subgraph induced by all vertices v with $e_k(v) = \mathrm{srad}_k(G)$. In regards to Figure 2.1, it is not hard to see that $\mathrm{sdiam}_3(G) = \mathrm{e}_3(A) = 7$, while $\mathrm{srad}_3(G) = \mathrm{e}_3(D) = 5$. The 3-center of G, $C_3(G)$ is composed solely of the vertex D. For a general graph, the following connection between the Steiner distance and the standard distance is immediate.

Observation 2.2. If G is a connected graph and $v \in V(G)$, then $\mathrm{e}_2(v) = \mathrm{e}(v)$, $\mathrm{srad}_2(G) = \mathrm{rad}(G)$, $\mathrm{sdiam}_2(G) = \mathrm{diam}(G)$, and $C_2(G) = C(G)$.

2.2 Steiner distance results generalizing standard distance results

Many classic results for the standard distance have been generalized for the Steiner distance. In this section, we provide a number of these generalizations. In 1989, Erdős, Pach, Pollack, and Tuza proved the following relationship between the diameter of a connected graph and its minimum degree.

Theorem 2.3. [23] If G is a connected graph of order n and minimum degree δ, then

$$\mathrm{sdiam}_2(G) \leq \frac{3n}{\delta + 1} + O(1).$$

Ten years later, this result was extended to the Steiner k-diameter by Dankelmann, Swart, and Oellermann.

Theorem 2.4. [16] Suppose that G is a connected graph of order n and minimum degree δ. Then, if $2 \leq k \leq n$ is an integer,

$$\text{sdiam}_k(G) \leq \frac{3n}{\delta + 1} + 3k.$$

Ali, Dankelmann, and Mukewmbi [4] later improved this bound and gave even better bounds for graphs which are triangle-free or contain no 4-cycles. In [16], Dankelmann, Swart, and Oellermann generalized even more diameter results to the Steiner k-diameter. In 1985, Harary and Robinson proved the following fact.

Theorem 2.5. [31] Suppose that G is a connected graph with complement $\bar{G}$. If $\text{sdiam}_2(G) \geq 3$, then $\text{sdiam}_2(\bar{G}) \leq 3$.

A generalized version for the Steiner k-diameter is as follows.

Theorem 2.6. [16] Let G be a connected graph with complement $\bar{G}$ and $\text{sdiam}_k(G) = k + \ell$ where $1 \leq \ell \leq k - 1$. Then

$$\text{sdiam}_k(G) + \text{sdiam}_k(\bar{G}) \leq 3k.$$

More recently, Mao [43] gave general bounds improving on those in Theorem 2.6 and obtained sharp results for $\text{sdiam}_3(G) + \text{sdiam}_3(\bar{G})$ and $\text{sdiam}_3(G) \cdot \text{sdiam}_3(\bar{G})$. We wish to include one final result proven by Dankelmann, Oellermann, and Swart, which can be used to bound $\text{sdiam}_k(G)$.

Theorem 2.7. [16] Suppose that G is a connected graph of order $|G| = n$ and $\text{sdiam}_k(G) = d_k$. Then

$$\|G\| \leq d_k + \binom{k-1}{2} + \binom{n - d_k - 1}{2} + (n - d_k - 1)(k + 1).$$

Furthermore, if G is 2-connected, then

$$\text{sdiam}_k(G) \leq \left\lfloor \frac{n(k-1)}{k} \right\rfloor.$$

Both these bounds are sharp.

The first of these results is an extension of a result of Ore [47] and can be used to find an upper bound on the Steiner k-diameter of G while the sharpness of the second bound is witnessed by the cycle with n vertices.

In 2013, Ali [3] examined the Steiner k-diameter in relation to the girth, length of a shortest cycle, of a graph.

Theorem 2.8. [3] Suppose that G is a connected graph of order n, girth g, and minimum degree $\delta \geq 3$. Let $2 \leq k \leq n$ be an integer.

1. If g is odd, then

$$\mathrm{sdiam}_k(G) \leq g\frac{n}{K} + (g-1)k - 2g + 1, \text{ where } K = 1 + \delta\frac{(\delta+1)^{(g-1)/2} - 1}{\delta - 2}.$$

2. If g is even, then

$$\mathrm{sdiam}_k(G) \leq g\frac{n}{L} + (g-1)k - 2g + 2, \text{ where } L = 2\delta\frac{(\delta-1)^{g/2} - 1}{\delta - 2}.$$

A graph parameter with applications to chemistry is the Wiener index. Given a connected graph G, the Weiner index $W(G)$ is the sum off all distances between all pairs of vertices. That is

$$W(G) = \sum_{\{u,v\} \subset V(G)} d(u,v).$$

The Wiener index was introduced in 1947 by Harold Wiener, a chemistry student, as a way to predict the boiling points of alkanes [45]. Since its inception, the Wiener index has been the subject of much mathematical attention (See for instance [11] and [42]). A related parameter is the average distance of the graph G, $\mu(G)$, which evaluates the average distance between all pairs of vertices in a graph. This is directly related to the Wiener index by the following equation:

$$\mu(G) = \frac{W(G)}{\binom{|V(G)|}{2}}.$$

Both of these parameters are easily generalized for the Steiner distance. Given an integer $k \geq 2$, the Steiner k-Wiener index, $W_k(G)$ is defined by

$$W_k(G) := \sum_{S \subset V(G), |S|=k} d(S).$$

Similarly, the Steiner k-average distance, $\mu_k(G)$, is defined by

$$\mu_k(G) := \frac{W_k(G)}{\binom{|V(G)|}{k}}.$$

In 1996, Dankelmann et al. [15] computed the following bounds for the Steiner k-average distance of a tree.

Theorem 2.9. [15] Suppose that G is a connected graph of order n and $2 \leq k \leq n$. Then,

$$k - 1 \leq \mu_k(G) \leq \frac{k-1}{k+1}(n+1).$$

Equality is achieved on the left if and only if G is $(n+1-k)$-connected and on the right if and only if G is a path or $n = k$.

Setting $k = 2$ in Theorem 2.9, this result generalizes the following result of Entringer, Jackson, and Snyder [21].

Theorem 2.10 ([21])**.** Suppose that G is a connected graph of order n. Then,

$$1 \leq \mu(G) \leq \frac{1}{3}(n+1).$$

Equality is achieved on the left if and only if $G = K_n$, the complete graph with n vertices and on the right if and only if G is a path.

In the same paper, the authors made the following conjecture relating the Steiner k-average distance to the standard average distance. The authors were able to confirm the conjecture for the case when $k = 3$.

Conjecture 2.11. [15] If G is a connected graph of order n and $2 \leq k \leq n$, then

$$\mu_k(G) \geq 3\frac{k-1}{k+1}\mu(G).$$

This inequality is true for $k = 3$.

2.3 A look at trees

Restricting to trees, still more can be said about the Steiner k-average distance.

Theorem 2.12. [15] Suppose that T is a tree of order $n \geq k$ and $2 \leq m \leq k-1$, then

$$\mu_k(T) \leq \frac{k}{m}\mu_m(T).$$

Equality holds if T is a star.

Given the complexity of finding a Steiner tree in a general graph, efficient algorithms for finding $SW_k(G)$ or $\mu_k(G)$ seem unlikely. Fortunately, in trees computing the Steiner k-Wiener index can be more easily completed. In fact, an algorithm was outlined in [15] showing that it is possible to compute $SW_k(T)$ in $O(nk)$ time.

This simplicity in computing Steiner parameters in trees lends itself to finding the Steiner k-center of a tree. The following algorithm was given by Oellermann in [46].

Algorithm 2.13. [46] To find the Steiner k-center of a tree, T, of order $n \geq k \geq 2$:

1. $H \leftarrow T$
2. If H has at most $k-1$ end-vertices (vertices of degree 1), or if $H \cong K_1$ or $H_2 \cong K_2$ and $k = 2$, output H as H is the Steiner k-center of T and stop; otherwise continue.
3. Delete all end-vertices from H and replace H with the resulting tree. Then return to step 2.

Figure 2.2 illustrates algorithm 2.13 on a tree with 5 leaves. Coupling with this algorithm, Oellermann proved the following theorem, which implies that all trees with at most $k-1$ leaves are the Steiner k-center of some tree.

Theorem 2.14. [46] Fix a tree H. There exists a tree T for which $H \cong C_k(T)$ if and only if one of the following hold:

1. $k \geq 3$ and H has at most $k-1$ leaves, or

2. $k = 2$ and $H \cong K_1$ or $H \cong K_2$.

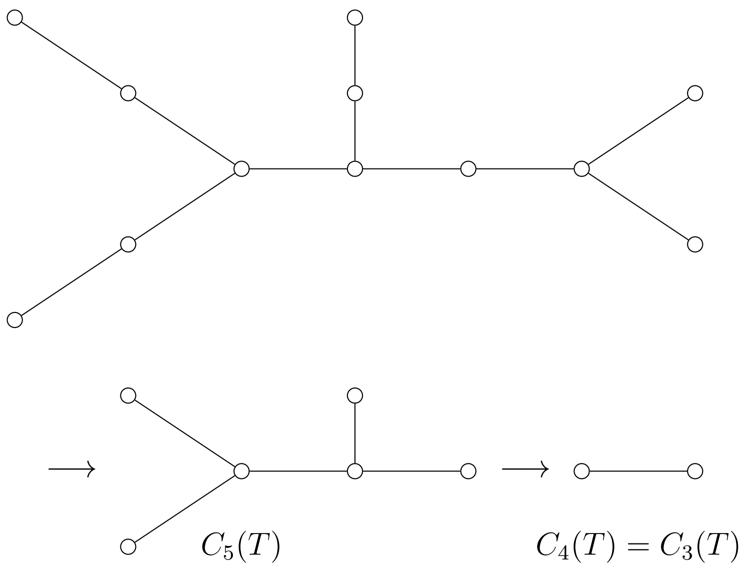

Figure 2.2 The Tree T with $C_5(T)$ and $C_4(T)$. For $k \geq 6$, $C_k(T) = T$.

For the remainder of this dissertation's content related to the Steiner distance in graphs, we will focus on the Steiner k-diameter and Steiner k-radius. A classic result for the standard distance relates the diameter to the radius.

Theorem 2.15. If G is a connected graph, then

$$\operatorname{diam}(G) \leq 2\operatorname{rad}(G).$$

This bound is tight as witnessed by a path of even length.

In [12], the following generalization was made for trees.

Theorem 2.16. [12] If T is a tree, then

$$\operatorname{sdiam}_k(T) \leq \frac{k}{k-1}\operatorname{srad}_k(T).$$

This bound is tight, as witnessed by a star with at least k leaves.

It is not hard to see that if T is a tree with less than k leaves, then $\mathrm{sdiam}_k(T) = \mathrm{srad}_k(T)$. We can also prove the following.

Proposition 2.17. Given integers k, r, and d, with $k \geq 2$ and $k-1 \leq r \leq d \leq \frac{k}{k-1}r$, one can find a tree T with $\mathrm{sdiam}_k(T) = d$ and $\mathrm{srad}_k(T) = r$.

While the proof of this result is simple enough, we have not found such a proof in the literature. For completeness, we include one here.

Proof. If $r = d$, let T be the path of length r. For $r < d$, we will construct a "spider," T, with k legs. Let a, b be integers such that $r = a(k-1) + b$ and $0 \leq b \leq k-2$. Construct T so that b legs are of length $a+1$, while $k-1-b$ legs are of length a, and one leg is of length $d-r$. An example of such a spider is illustrated in Figure 2.3 for $k = 4$, $d = 12$, and $r = 10$. We calculate $a = 3$ and $b = 1$. Hence, we have $b = 1$ leg of length 4, $k-1-b = 2$ legs of length 3, and one leg of length $d - a = 2$.

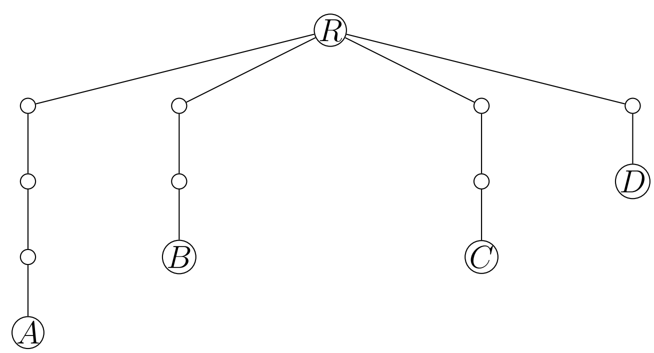

Figure 2.3 A spider T with $\mathrm{sdiam}_4(T) = 12$ and $\mathrm{srad}_4(T) = 10$.

Since the constructed tree has k leaves, $\mathrm{sdiam}_k(T) = \|T\| = d$. To show that $\mathrm{srad}_k(T) = r$, we use the fact (proven in [12]) that $\mathrm{srad}_k(T) = \mathrm{sdiam}_{k-1}(T)$. To maximize $d(S)$ over all $|S| = k-1$, we consider the pendant vertices on the $k-1$ longest legs of T. Hence, $\mathrm{sdiam}_{k-1}(T) = r$ and $\mathrm{srad}_k(T) = r$. $\square$

For general connected graphs, Theorem 2.16 does not hold for $k \geq 3$ as shown in [32]. In Chapter 3, we find a tight upper bound for the ratio $\operatorname{sdiam}_k(G)/\operatorname{srad}_k(G)$ for any connected graph G and $k \geq 5$.

Chapter 3

The Steiner diameter and Steiner Radius in General Graphs

3.1 Bounding the Steiner diameter with respect to the Steiner radius

Restating Theorem 2.15 in the context of the Steiner k-radius and Steiner k-diameter, we have that for any connected graph G,

$$\mathrm{sdiam}_2(G) \leq 2\,\mathrm{srad}_2(G).$$

After proving Theorem 2.16, the authors of [12] conjectured their result for all connected graphs.

Conjecture 3.1. [12] Let $k \geq 2$ be an integer and G be a connected graph of order $n \geq k$. Then,

$$\mathrm{sdiam}_k(G) \leq \frac{k}{k-1}\,\mathrm{srad}_k(G).$$

Soon after, this conjecture was proven incorrect. For each $k \geq 3$, the authors of [32] constructed an infinite family of graphs satisfying $\mathrm{sdiam}_k(G) = \frac{2(k+1)}{2k-1}\,\mathrm{srad}_k(G)$. The simplest member of the family for $k = 4$ is illustrated in Figure 3.11 by the graph H_4. In the same paper, the following conjecture was made.

Conjecture 3.2. [32] Suppose that G is a connected graph with order at least k. Then

$$\mathrm{sdiam}_k(G_k) \leq \frac{2(k+1)}{2k-1}\,\mathrm{srad}_k(G_k).$$

Furthermore, the authors provided proofs of the inequality for $k = 3, 4$. However, their proof for $k = 4$ was incorrect.

We break this chapter into the following divisions. In section 3.2, we make necessary definitions and prove some preliminary lemmas required for our main results. In section 3.3, we provide a correct proof to confirm the conjecture in [32] for $k = 4$, showing the following:

Theorem 3.3. If G is a connected graph of order at least 4, then

$$\operatorname{sdiam}_4(G) \leq \frac{10}{7} \operatorname{srad}_4(G).$$

This bound was shown to be tight in [32]. In section 3.5, we prove our main result:

Theorem 3.4. If G is a connected graph and $k \geq 5$ is an integer, then

$$\operatorname{sdiam}_k(G) \leq \frac{k+3}{k+1} \operatorname{srad}_k(G).$$

In section 3.5, we show that this bound is tight for each $k \geq 5$. It is worth noting that the bound in Conjecture 3.2 matches that of Theorem 3.4 if $k = 5$. In section 3.6 we will identify the error in [32]. To summarize, table 3.1 gives the maximum value of the ratio $\operatorname{sdiam}_k(G)/\operatorname{srad}_k(G)$ for a connected graph G.

Table 3.1 Values of $\operatorname{sdiam}_k(G)/\operatorname{srad}_k(G)$ as found in [32] and this paper.

k	$\operatorname{sdiam}_k(G)/\operatorname{srad}_k(G)$	Reference
3	8/5	[32]
4	10/7	[32] and Section 3.3
≥ 5	$(k+3)/(k+1)$	Section 3.4

3.2 Definitions and Preliminary Lemmas

Let $k \geq 2$ be a positive integer and suppose that G is a connected graph of order at least k. Then there exists a set $D = \{v_1, v_2, \ldots, v_k\} \subset V(G)$ such that

$d(D) = \mathrm{sdiam}_k(G)$. Similarly, there exists $v_0 \in V(G)$ satisfying $\mathrm{e}_k(v_0) = \mathrm{srad}_k(G)$. We may now make the following definitions, which closely follow definitions made in [32].

Definition 3.5. Suppose that G is a connected graph of order at least k. Assume that $D = \{v_1, v_2, \ldots, v_k\}$ with $d(D) = \mathrm{sdiam}_k(G)$ and $\mathrm{e}_k(v_0) = \mathrm{srad}_k(G)$. For each $1 \le i \le k$,

1. Define $D_i := (D \setminus \{v_i\}) \cup \{v_0\}$;

2. Define T_i to be a Steiner tree for D_i;

3. Define T_i' to be the smallest subtree of T_i spanning $D_i \setminus \{v_0\}$;

4. Define $\ell_i := \|T_i\| - \|T_i'\|$. Without loss of generality, we assume that $\ell_1 \le \ell_j$ for $j \ge 2$.

Of course, if $v_0 \in D$, we have that $\mathrm{srad}_k(G) = \mathrm{sdiam}_k(G)$. So if $\mathrm{srad}_k(G) < \mathrm{sdiam}_k(G)$ we must have $v_0 \notin D$. It is worth noting that v_i is the only element of $D \cup \{v_0\}$ not necessarily contained in the tree T_i, while the tree T_i' need not contain v_0. Figure 3.1 illustrates the difference between the trees T_1 and T_1' for $k = 3$.

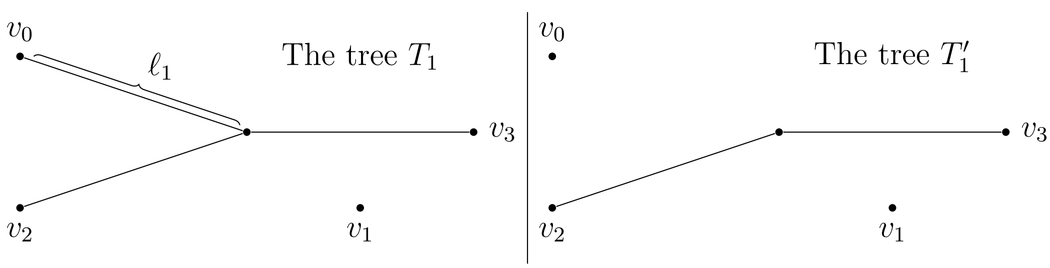

Figure 3.1 Possible trees T_1 and T_1' for $k = 3$. Vertices of degree 2 are omitted.

Note that $\ell_i = d_{T_i}(v_0, T_i')$. From Definition 3.5, we make the following observation.

Observation 3.6. Suppose that $k \geq 2$ is an integer and that G is a connected graph with at least k vertices. Let ℓ_i, T_i, and T_i' be defined as in Definition 3.5. If $\text{sdiam}_k(G) > p\,\text{srad}_k(G)$ for some $p > 0$, then for each $1 \leq i \leq k$, we have the following:

1. $\|T_i\| \leq \text{srad}_k(G) < \frac{1}{p}\text{sdiam}_k(G)$, and
2. $\|T_i'\| = \|T_i\| - \ell_i < \frac{1}{p}\text{sdiam}_k(G) - \ell_1$.

With Observation 3.6 in mind, we now prove our first lemma.

Lemma 3.7. Suppose that G is a connected graph of order $n \geq k$. Let ℓ_i, T_i, and T_i' be defined as in Definition 3.5. If $\text{sdiam}_k(G) > p\,\text{srad}_k(G)$ with $p > 1$, then for $1 \leq i, j \leq k$ with $i \neq j$, the following hold:

1. $d_{T_1}(v_i, v_0) > \dfrac{p-1}{p}\text{sdiam}_k(G)$, and
2. $d_{T_1}(v_i, v_j) > \dfrac{p-1}{p}\text{sdiam}_k(G) + \ell_1$.

Proof. For the first inequality, note that adjoining the tree T_i with the path in T_1 between v_i and v_0 generates a connected subgraph of G spanning D. Hence,

$$\|T_i\| + d_{T_1}(v_i, v_0) \geq \text{sdiam}_k(G),$$

which implies that

$$d_{T_1}(v_i, v_0) \geq \text{sdiam}_k(G) - \|T_i\|.$$

In view of Observation 3.6, we see that

$$\begin{aligned} d_{T_1}(v_i, v_0) &> \text{sdiam}_k(G) - \left(\frac{1}{p}\text{sdiam}_k(G)\right) \\ &= \frac{p-1}{p}\text{sdiam}_k(G). \end{aligned}$$

For the second inequality, we similarly note that adjoining the tree T_i' with the path in T_1 between v_i and v_j generates a connected subgraph of G spanning D. Hence,

$$\|T_i'\| + d_{T_1}(v_i, v_j) \geq \text{sdiam}_k(G),$$

which implies that

$$d_{T_1}(v_i, v_j) \geq \mathrm{sdiam}_k(G) - \|T_i'\|.$$

Applying Observation 3.6 a second time, we have that

$$\begin{aligned} d_{T_1}(v_i, v_j) &> \mathrm{sdiam}_4(G) - \left(\frac{1}{p}\,\mathrm{sdiam}_k(G) - \ell_i\right) \\ &= \frac{p-1}{p}\,\mathrm{sdiam}_k(G) + \ell_i \\ &\geq \frac{p-1}{p}\,\mathrm{sdiam}_k(G) + \ell_1. \end{aligned}$$

□

With Lemma 3.7 in hand, we make the following observation.

Corollary 3.8. Using the definitions and notation provided in Definition 3.5, if $1 < i \neq j \leq k$ and

$$\mathrm{sdiam}_k(G) > \frac{10}{7}\,\mathrm{srad}_k(G),$$

then

1. $d_{T_1}(v_i, v_0) > \frac{3}{10}\,\mathrm{sdiam}_k(G)$, and
2. $d_{T_1}(v_i, v_j) > \frac{3}{10}\,\mathrm{sdiam}_k(G) + \ell_1$.

Furthermore, if $1 < i \neq j \leq k$ and

$$\mathrm{sdiam}_k(G) > \frac{k+3}{k+1}\,\mathrm{srad}_k(G),$$

then

1. $d_{T_1}(v_i, v_0) > \frac{2}{k+3}\,\mathrm{sdiam}_k(G)$, and
2. $d_{T_1}(v_i, v_j) > \frac{2}{k+3}\,\mathrm{sdiam}_k(G) + \ell_1$.

With these definitions and results in hand, we are prepared to prove our main results.

3.3 Proof of Theorem 3.3

Proof. Suppose towards contradiction that there exists a graph G satisfying

$$\mathrm{sdiam}_4(G) > \frac{10}{7}\,\mathrm{srad}_4(G).$$

This implies that

$$\mathrm{srad}_4(G) < \frac{7}{10}\,\mathrm{sdiam}_4(G). \tag{3.1}$$

Suppose that $D = \{v_1, v_2, v_3, v_4\}$ is a set of vertices in G such that $d(D) = \mathrm{sdiam}_4(G)$ and $v_0 \in C_4(G)$. For $1 \le i \le 4$, define D_i, T_i, T_i', and ℓ_i as in Definition 3.5. Again, we assume that $\ell_1 \le \ell_j$ for $j \ge 2$.

We first consider the cases where T_1 is a path or a subdivision of the star on 3 vertices. These cases were correctly covered in [32]. We include them here for completeness.

First, suppose that T_1 is a path. Relabel the elements of D_1 as u_1, u_2, u_3 and u_4 so that the tree T_1 is a concatenation of paths $u_1 - u_2 - u_3 - u_4$. See Figure 3.2 for an illustration of this situation.

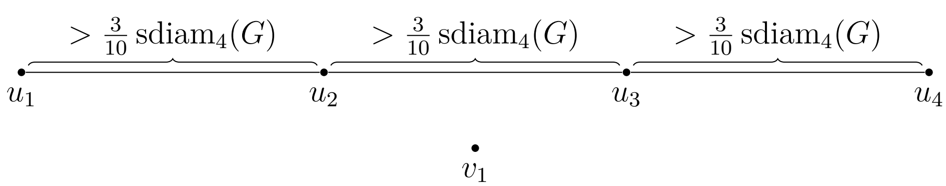

Figure 3.2 The tree T_1 as a path. Vertices of degree two not in D_1 are omitted.

Now, T_1 is composed of three paths between elements of D_i. By Corollary 3.8, each of these paths has length at least $\frac{3}{10}\,\mathrm{sdiam}_4(G)$. So

$$\mathrm{srad}_4(G) \ge ||T_1|| > \frac{9}{10}\,\mathrm{sdiam}_4(G),$$

which contradicts equation (3.1).

Next, we suppose T_1 has exactly three leaves. Label them as u_1, u_2, and u_3. Let u_4 be the element of D_1 which is an interior vertex of T_1 and let s be the vertex of degree 3 in T_1. It is possible that $s = u_4$. Without loss of generality, suppose that u_4 lies on the $s - u_3$ path in T_1. Define the following distances as illustrated in Figure 3.3.

$$a := d_{T_1}(u_1, s) \quad b := d_{T_1}(u_2, s)$$
$$c := d_{T_1}(u_3, u_4) \quad d := d_{T_1}(u_4, s).$$

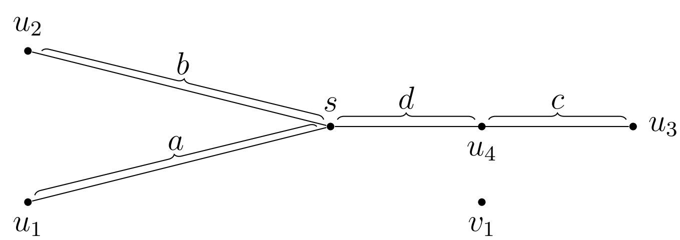

Figure 3.3 The tree T_1 with only three leaves.
Vertices of degree two not in D_1 are omitted.

Consider the following sum:

$$(a + b) + (a + d) + (b + d) + 2c = 2a + 2b + 2c + 2d.$$

The right hand side of this equation counts each edge of T_1 twice. Hence, by equation (3.1),

$$2a + 2b + 2c + 2d = 2\|T_1\| \le 2\,\mathrm{srad}_4(G) < \frac{14}{10}\,\mathrm{sdiam}_4(G). \tag{3.2}$$

But Corollary 3.8 implies that the left hand side of the equation is bounded below by

$$\begin{aligned}(a + b) + (a + d) + (b + d) + 2c &\ge 5 \cdot \min\{d_{T_1}(u_i, u_j) : 1 \le i \ne j \le 4\}\\ &> 5 \cdot \frac{3}{10}\,\mathrm{sdiam}_4(G)\\ &= \frac{15}{10}\,\mathrm{sdiam}_4(G),\end{aligned}$$

which contradicts equation (3.2).

We now suppose that T_1 has exactly 4 leaves. We note that T_1 has at most two vertices of degree at least 3. Let s be the vertex of degree at least 3 in T_1 closest to v_0. Relabel the leaves of T_1 as $\{v_0, u_1, u_2, u_3\}$ so that s is the nearest neighbor (in T_1) of degree at least 3 to u_3 as well. Next, let t be the vertex of degree at least 3 in T_1 which is closest to u_2 (in T_1). It is possible that $s = t$. Figure 3.4 illustrates this situation.

By Definition 3.5, we have that ℓ_1 is the distance between v_0 and s in T_1. Define the following distances as illustrated in Figure 3.4:

$$a := d_{T_1}(u_1, t) \quad b := d_{T_1}(u_2, t)$$
$$c := d_{T_1}(u_3, s) \quad d := d_{T_1}(s, t).$$

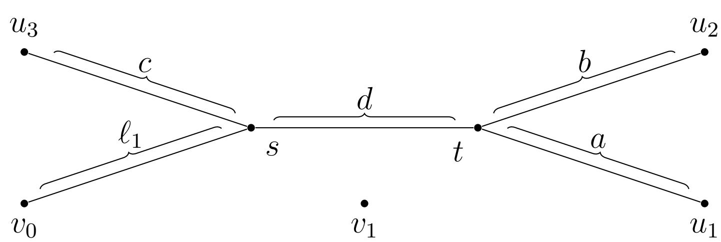

Figure 3.4 The tree T_1 and the vertex v_1. Vertices of degree 2 are omitted.

We now consider the sum

$$2(\ell_1 + a + b + c + d) = (\ell_1 + c) + (c + d + b) + (a + b) + (\ell_1 + d + a).$$

By Corollary 3.8, the left hand side is bounded below by

$$(\ell_1 + c) + (c + d + b) + (a + b) + (\ell_1 + d + a) > \frac{12}{10}\,\mathrm{sdiam}_4(G) + 2\ell_1,$$

while, as in the previous case, by equation (3.1), we have that the right hand side is bounded below by

$$2(\ell_1 + a + b + c + d) = 2\|T_i\| < \frac{14}{10}\,\mathrm{sdiam}_4(G).$$

Combining these inequalities together, we have that

$$\frac{12}{10}\,\mathrm{sdiam}_4(G) + 2\ell_1 < \frac{14}{10}\,\mathrm{sdiam}_4(G),$$

which implies that

$$\ell_1 < \frac{1}{10}\,\mathrm{sdiam}_4(G). \tag{3.3}$$

Alternatively, we may consider the sum

$$2\ell_1 + 2(\ell_1 + a + b + c + d) = (\ell_1 + d + b) + (\ell_1 + d + a) + 2(\ell_1 + c) + (a + b).$$

Applying corollary 3.8, we see that

$$(\ell_1 + d + b) + (\ell_1 + d + a) + 2(\ell_1 + c) + (a + b) > \frac{15}{10}\,\mathrm{sdiam}_4(G) + \ell_1.$$

But by equation (3.1), we have that

$$2\ell_1 + 2(\ell_1 + a + b + c + d) < \frac{14}{10}\,\mathrm{sdiam}_4(G) + 2\ell_1.$$

Combining these inequalities together, we see that

$$\frac{15}{10}\,\mathrm{sdiam}_4(G) + \ell_1 < \frac{14}{10}\,\mathrm{sdiam}_4(G) + 2\ell_1,$$

which implies that $\ell_1 > \frac{1}{10}\,\mathrm{sdiam}_4(G)$, a contradiction of equation (3.3). □

3.4 Proof of Theorem 3.4

Proof. Suppose towards contradiction that G is a connected graph with

$$\mathrm{sdiam}_k(G) > \frac{k+3}{k+1}\,\mathrm{srad}_k(G).$$

This implies that

$$\mathrm{srad}_k(G) < \frac{k+1}{k+3}\,\mathrm{sdiam}_k(G). \tag{3.4}$$

Suppose $D = \{v_1, v_2, \ldots, v_k\}$ is a set of vertices such that $d(D_k) = \mathrm{sdiam}_k(G)$. Let $v_0 \in C_k(G)$. For $1 \leq i \leq k$, define D_i, T_i, T_i', and ℓ_i as in Definition 3.5. Again, we assume that $\ell_1 \leq \ell_j$ for $j \geq 2$. We have that $\|T_1\| \leq \text{srad}_k(G)$. Let x be the vertex in T_1', which is closest to v_0 in T_1. It is possible that $x = v_0$. We now root T_1 at v_0 and consider the following two cases.

Case 1: $x \in D_i \setminus \{v_0\} = \{v_2, \ldots, v_k\}$.

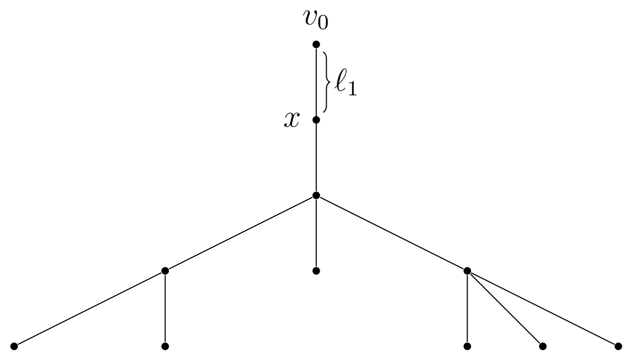

Figure 3.5 A possible picture of the tree T_1 in case 1. Unnamed vertices of degree 2 are omitted.

Since $x \in D_i$, we have that $d_{T_1}(v_0, x) = d_{T_1}(v_0, v_i)$ for some $2 \leq i \leq k$. Then, by Corollary 3.8, we have $\ell_1 > \frac{2}{k+3} \text{sdiam}_k(G)$. Traversing T_1 via a depth first search and returning to v_0 induces a new labeling of the elements of D_1 in the following way: Let $u_1, u_2, \ldots, u_{k-1}$ be a relabeling of the vertices $v_2, \ldots, v_k$ in the order in which these vertices are visited first in the depth first search. By corollary 3.8, we have that $d_{T_1}(v_0, u_1) > \frac{2}{k+3} \text{sdiam}_k(G)$ and $d_{T_1}(v_0, u_{k-1}) > \frac{2}{k+3} \text{sdiam}_k(G)$. Furthermore, corollary 3.8 asserts that $d(u_i, u_j) > \frac{2}{k+3} \text{sdiam}_k(G) + \ell_1$. Since $\ell_1 > \frac{2}{k+3} \text{sdiam}_k(G)$, the length of this traversal is greater than

$$\begin{aligned}
&2 \cdot \frac{2}{k+3} \text{sdiam}_k(G) + (k-2)\left(\frac{2}{k+3} + \ell_1\right) \\
&> 2 \cdot \frac{2}{k+3} \text{sdiam}_k(G) + (k-2)\left(\frac{2}{k+3} + \frac{2}{k+3}\right) \\
&= \frac{4(k-1)}{k+3} \text{sdiam}_k(G).
\end{aligned}$$

This traversal also visits each edge of T_1 exactly twice, which implies that

$$2\,\mathrm{srad}_k(G) \geq 2\|T_1\| > \frac{4(k-1)}{k+3}\,\mathrm{sdiam}_k(G).$$

Since $k \geq 5$, we have contradicted equation (3.4).

Case 2: $x \notin D_i \setminus \{v_0\}$.

Since $x \notin D_i \setminus \{v_0\}$, we have that x has at least 2 children. Pick a child of x, say c. Let H_1 be the tree induced by vertices of the v_0c path and descendants of c, and let H_2 be the tree obtained from T_1 by removing c and its descendants. Figure 3.6 illustrates the differences between T_1, H_1, and H_2.

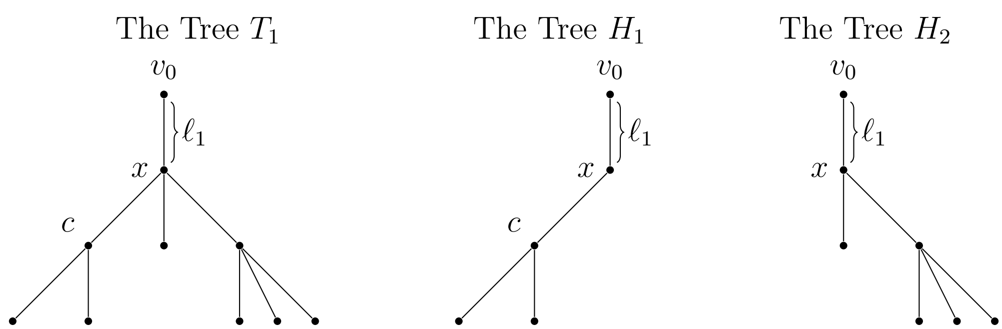

Figure 3.6 A possible picture of T_1, H_1, and H_2 as in case 2. Unnamed vertices of degree 2 are omitted.

Both H_1 and H_2 contain elements of D_i. We observe that $E(H_1) \cup E(H_2) = E(T_1)$ while the intersection of $E(H_1)$ and $E(H_2)$ is the path in T_1 between v_0 and x. Hence,

$$\|H_1\| + \|H_2\| - \ell_1 = \|T_1\| < \frac{k+1}{k+3}\,\mathrm{sdiam}_k(G). \tag{3.5}$$

It is easy to see that $|V(H_1) \cap D_1| + |V(H_2) \cap D_1| = k+1$ since v_0 (and only v_0) is included in both subtrees. As in the previous case, we root H_1 and H_2 at v_0 and perform a depth first search traversal of each subtree. By the same reasoning as the previous case, we see that

$$2\|H_1\| > |V(H_1) \cap D_1| \cdot \frac{2}{k+3}\,\mathrm{sdiam}_k(G) + (|V(H_1) \cap D_1| - 2)\ell_1,$$

and

$$2\|H_2\| > |V(H_2) \cap D_1| \cdot \frac{2}{k+3} \operatorname{sdiam}_k(G) + (|V(H_2) \cap D_1| - 2)\ell_1.$$

Combining these sums together, we see that

$$2\|H_1\| + 2\|H_2\| > (k+1) \cdot \frac{2}{k+3} \operatorname{sdiam}_k(G) + (k+1-4)\ell_1.$$

Since $k \geq 5$, we have that

$$2\|H_1\| + 2\|H_2\| > \frac{2(k+1)}{k+3} \operatorname{sdiam}_k(G) + 2\ell_1.$$

Hence,

$$\begin{aligned} 2\|H_1\| + 2\|H_2\| - 2\ell_1 &> \frac{2(k+1)}{k+3} \operatorname{sdiam}_k(G) \\ \|H_1\| + \|H_2\| - \ell_1 &> \frac{k+1}{k+3} \operatorname{sdiam}_k(G), \end{aligned}$$

which contradicts equation (3.5). □

3.5 Sharpness of Theorem 3.4

We now prove that this bound in Theorem 3.4 is tight via a construction. Let $k \geq 5$ be an integer. We now outline the construction of a graph G_k satisfying

$$\operatorname{sdiam}_k(G_k) = \frac{k+3}{k+1} \operatorname{srad}_k(G_k).$$

Begin with a set of k independent vertices, $D = \{d_1, d_2, \ldots, d_k\}$. Let $m = \lceil \frac{k+1}{2} \rceil$. Define $D_1 = \{d_1, d_2, \ldots, d_m\}$ and $D_2 = \{d_m, d_{m+1}, \ldots, d_k\}$. For each vertex $d_i \in D_1$, adjoin to each vertex in $D_1 \setminus \{d_i\}$ a new vertex a_i. Let A be the set these new vertices all such vertices. Similarly, for each vertex $d_j \in D_2$ define a new vertex b_j to be a vertex with $N(b_u) = D_2 \setminus \{d_j\}$. Let B be the set of all such vertices. Finally, adjoin a new vertex r to each vertex in $A \cup B$. This completes the construction of G_k. Figures 3.7 and 3.8 illustrate the graphs of G_5 and G_6, respectively.

We now show that $\operatorname{sdiam}_k(G_k) = k+3$ and $\operatorname{sdiam}_k(G_k) = k+1$ via a series of three claims. This proves that the bound in Theorem 3.4 is tight for each $k \geq 5$.

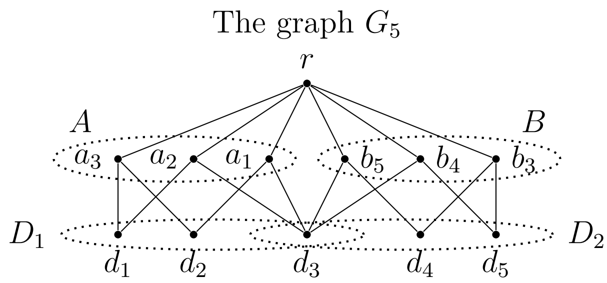

Figure 3.7 The graph G_5. All vertices are shown.

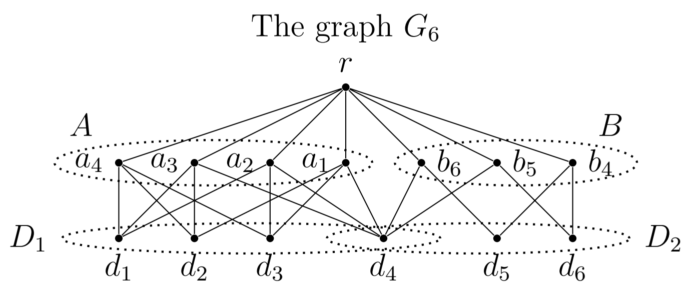

Figure 3.8 The graph G_6. All vertices are shown.

Claim 3.9. In the graph G_k, $d(D) = k + 3$.

Proof. Let T be a Steiner tree of D. Since T spans D, each element of D is incident to at least one edge in T, so there must be at least k such edges. Let E_1 be a set of k edges of T obtained by selecting precisely one edge incident to each d_i. Then the edges of E_1 induce a subgraph of G_k whose components are stars with centers in $A \cup B$. As for any $u, v \in A \cup B$, $|\nabla u \cup \nabla v| \leq k - 1$, we have that $G_k[E_1]$, the subgraph of G_k induced by E_1, has at least 3 connected components.

If $G_k[E_1]$ contains strictly more than 3 connected components, then at least 3 edges are required to connect these components, which implies that $\|T\| \geq k + 3$. Suppose then that $G_k[E_1]$ contains exactly 3 connected components, which are stars centered on the 3 vertices x, y, z in $A \cup B$ (we denote them by S_x, S_y, S_z respectively). We label them so x is the vertex such that $xd_m \in E_1$. Without loss of generality $x \in A$, i.e. $x = a_j$ for some $1 \leq j < m$ (the case when $x \in B$ follows similarly). As one edge in E_1 is incident upon d_j we may assume it is yd_j and therefore $y \in A$. Then, the elements of $D_2 \setminus \{d_m\}$ must be contained in S_z, so $z = b_m$.

Observe that $d_{G_k}(S_x, S_z) = d_{G_k}(S_y, S_z) = 2$ and consider the set E_2 of edges of T that are not in E_1. As T is connected, the edges of E_2 connect S_z to at least one of S_x, S_y, so $|E_2| \geq 2$. If $|E_2| \geq 3$, then $||T|| \geq k+3$. If $E_2 = 2$, then we need at least one more edge for all three stars to be connected, so $||T|| \geq k+3$ in this case as well.

To show that $\|T\| = k+3$, consider the tree induced by the edge set

$$\{a_m d_i : 1 \leq i \leq m-1\} \cup \{b_m d_i : m+1 \leq i \leq k\} \cup \{d_{m-1}a_1, a_1 d_m, d_m b_1, b_1 d_k\}.$$

An illustration of the Steiner tree of D constructed above is included in Figure 3.9 for the case $k = 5$. This tree contains exactly $(m-1) + (k-m) + 4 = k+3$ edges and spans D. □

Note that Claim 3.9 implies that $\mathrm{e}_k(d_i) \geq k+3$ for each $1 \leq i \leq k$. We now move to showing that $\mathrm{e}_k(G) = k+1$.

Claim 3.10. In the graph G_k, we have that $\mathrm{e}_k(r) = k+1$.

Proof. Let $S \subset V(G_k)$ with $r \in V(G_k)$ and $|S| = k$. Suppose $s = |S \cap D|$. Since $s \leq k-1$, we have that $(S \cap D) \subset (\nabla a \cup \nabla b)$ for some $a \in A$ and $b \in B$. Then, if we consider the subgraph induced by the vertex set

$$S = (S \cap D) \cup \{a, b\} \cup (S \setminus D).$$

With $|S \cap D|$ edges, we connect vertices $S \cap D$ to $\{ab\}$ forming stars S_a, S_b. Adding the edges ra, rb connects S_a and S_b to a connected subgraph H. Then, with $k-1-s$ edges, we connect the elements of $S \setminus (D \cup \{r\})$ to H. In total, we have used $k-1+2 = k+1$ edges to connect the elements of S. Hence, $\mathrm{e}_k(r) \leq k+1$.

To show equality, consider the set $V_1 = \{d_1, d_2, \ldots, d_{k-1}, r\}$. Any tree spanning V_1 must contain at least $k-1$ edges between D and $A \cup B$. These $k-1$ edges induce at least 2 stars. These stars must be connected to r, which requires at least 2 edges. So any Steiner tree for V_1 contains at least $k-1+2 = k+1$ edges. Such a spanning tree for V_1 in the case $k=5$ is illustrated in Figure 3.9. $\square$

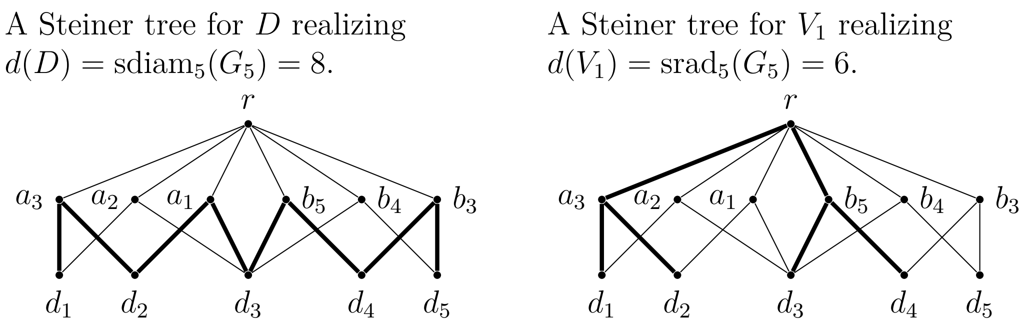

Figure 3.9 Steiner trees in the graph G_5 for D as in Definition 3.5 and the vertex set V_1 as above. The Steiner trees are in bold.

It should be stated that we have proven sufficient results to show that $\mathrm{srad}_k(G_k) = k+1$ and $\mathrm{sdiam}_k(G_k) = k+3$. Indeed, we have that $\mathrm{srad}_k(G_k) \leq \mathrm{e}_k(r) = k+1$ and $\mathrm{sdiam}_k(G_k) \geq k+3$. By Theorem 3.4, we can then infer that $\mathrm{sdiam}_k(G_k) = k+3$ and $\mathrm{srad}_k(G_k) = k+1$. For completeness, however, we will supply a proof requiring slightly more elbow grease. To do so, we need one more claim.

Claim 3.11. In the graph G_k suppose that $v \in A \cup B$. We have that $\mathrm{e}_k(v) = k+2$.

Proof. Suppose that v is an arbitrary element of $A \cup B$. By Claim 3.10, we have that $\mathrm{e}_k(r) = k+1$. Let $U_1 \subset V(G_k)$ be a vertex set of order k containing v. We may span U_1 by connecting a spanning tree for $(U_1 \cup \{r\}) \setminus \{v\}$ with the edge vr. Hence,

$$d(U_1 \cup \{r\}) \leq d((U_1 \cup \{r\}) \setminus \{v\}) + 1 \leq \mathrm{e}_k(r) + 1 = k+2.$$

This implies that, $\mathrm{e}_k(v) \leq k+2$.

We now prove equality. Suppose towards contradiction that $\mathrm{e}_k(v) < k+2$. Let $d_i \in D \cap N(v)$ and consider the vertex set $D^* = (D \setminus \{d_i\}) \cup \{v\}$. Since $\mathrm{e}_k(v) < k+2$ we have that $d(D^*) < k+2$. Then, joining the edge vd_i to a Steiner tree for D^* creates a subgraph spanning D with less than $k+3$ edges. We know such an edge exists since $d_i \in N(v)$. This contradicts Claim 3.9. Hence, $\mathrm{e}_k(v) = k+2$. □

With Claims 3.9, 3.10, and 3.11 in hand, we can prove the following proposition.

Proposition 3.12. For $k \geq 5$ the graph G_k satisfies $\mathrm{sdiam}_k(G_k) = k+3$ and $\mathrm{srad}_k(G_k) = k+1$.

Proof. Let $v \in A \cup B$. By Claims 3.10 and 3.11, we have that that $\mathrm{e}_k(r) = k+1$ and $\mathrm{e}_k(v) = k+2$, respectively. Now the only vertex set of size k which does not contain elements of $A \cup B \cup \{r\}$ is D. By Claim 3.9, we have that $d(D) = k+3$. Hence,

$$\mathrm{srad}_k(G) = \mathrm{e}_k(r) = k+1 \text{ and } \mathrm{sdiam}_k(G) = \mathrm{e}_k(d_1) = k+3.$$

□

3.6 Examining a previous proof

We now identify an error in the proof provided in [32] that for any connected graph G, $\mathrm{sdiam}_4(G) \leq \frac{10}{7}\mathrm{srad}_4(G)$. Given such a connected graph G, let $D = \{v_1, v_2, v_3, v_4\}$, v_0, D_i and T_i be defined as in Definition 3.5. For each $1 \leq i \leq k$, let s_i be the vertex of T_i with degree at least 3 in T_i closest to v_0 in T_i. For each $1 \leq i \leq 4$, we introduce a labeling of the leaves of T_i. For each such i, label $u_3^{(i)}$ be such that s_i us the nearest neighbor (in T_1) of degree 3 to $u_3^{(i)}$ as well. Next, let t_i be the vertex of degree at least 3 in T_1 which is closes to $u_2^{(i)}$ and $u_1^{(i)}$ (in T_1). It is possible that $s_i = t_i$. Then, for $1 \leq i \leq 4$, each tree T_i is of the form illustrated in Figure 3.10.

We then define the following distances as illustrated in Figure 3.10.

$$a_i := d_{T_1}(u_1, t) \quad b_i := d_{T_1}(u_2, t)$$

$$c_i := d_{T_1}(u_3, s) \quad d_i := d_{T_1}(s, t).$$

The authors of [32] define T_i'' to be the subtree of T_i obtained by deleting the vertices in the $u_3^{(i)} - s_i$ path except for s. Figure 3.10 illustrates the difference between T_i and T_i''.

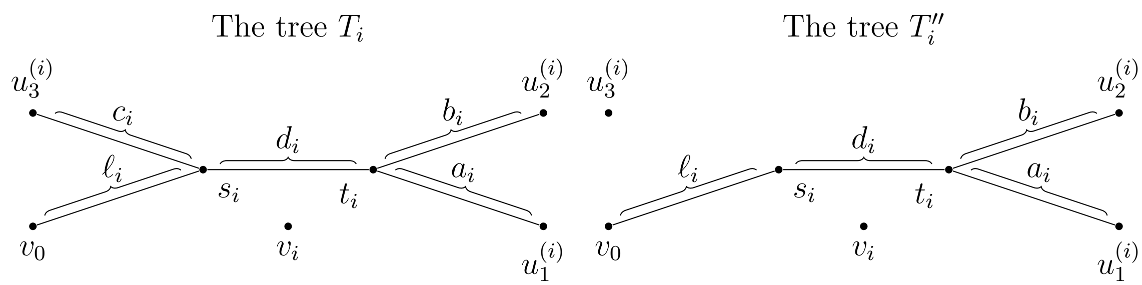

Figure 3.10 The trees T_i and T_i''. Vertices of degree 2 are omitted.

Suppose that T_{u_1} and T_{u_2} are minimal subtrees spanning $(D \setminus \{u_1\}) \cup \{v_0\}$ and $(D \setminus \{u_2\}) \cup \{v_0\}$, respectively. The authors make the following claims:

Claim 3.13. [32] In reference to Figure 3.10,

$$\|T_1''\| + a_4 + b_4 \geq \operatorname{sdiam}_4(G)$$

$$\|T_1''\| + a_4 + d_4 + \ell_4 \geq \operatorname{sdiam}_4(G)$$

$$\|T_3''\| + b_4 + d_4 + \ell_4 \geq \operatorname{sdiam}_4(G).$$

We now show that the first of these claims can be violated. Consider the graph H_4 given in Figure 3.11. Constructed in [32], the graph H_4 satisfies $\operatorname{sdiam}_4(H_4) = \frac{10}{7}\operatorname{srad}_k(G_4)$. Let $D = \{v_1, v_2, v_3, v_4\}$. It is easy to verify that $d(D) = 10$ and $v_0 \in C_4(G_4)$ with $\mathrm{e}_4(v_0) = 7$. Consider the sets $D_1 = \{v_0, v_2, v_3, v_4\}$ and $D_4 = \{v_0, v_1, v_2, v_3\}$. Let T_1 and T_4 be minimal spanning trees for D_1 and D_4, respectively. These trees are illustrated in Figure 3.12.

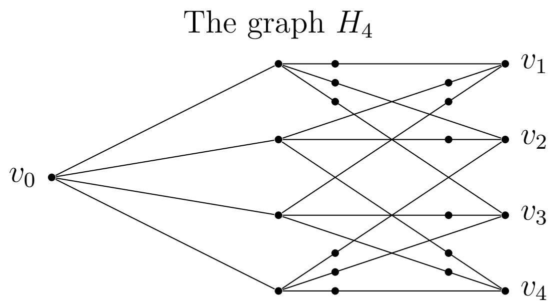

Figure 3.11 The graph H_4. All vertices are shown.

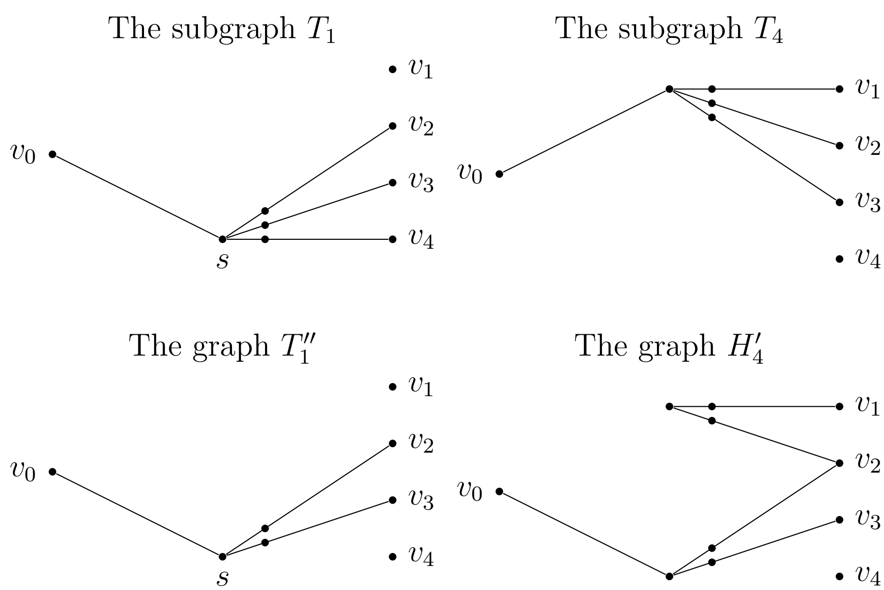

Figure 3.12 The pertinent subgraphs of H_4. All vertices are shown.

Let H_4' be the graph formed by adjoining to T_1'' the $v_1 - v_2$ path in T_4. This path corresponds to the $u_1^{(4)} - u_2^{(4)}$ path of length $a_4 + b_4$ in Figure 3.10. These graphs are illustrated in Figure 3.12. It is easy to verify that $\|H_4'\| = 9$, contradicting Claim 3.13. The remaining statements of Claim 3.13 can similarly be contradicted by the graph H_4.

Chapter 4

Steiner Distance of Large Sets in the Hypercube

4.1 Introduction

In addition to bounds of the k-Steiner diameter upon other properties of a graph, several results and bounds have been produced for specific graph classes. For the general connected graph, very general sharp bounds are known.

Theorem 4.1. [12] Suppose that k, n are integers with $2 \leq k \leq n$ and G is a connected graph of order n. Then,

$$k - 1 \leq \text{sdiam}_k(G) \leq n - 1.$$

Restricting to maximally planar graphs, Ali, Mukwembi, and Dankelmann [5] showed the following bounds for Steiner k-diameter of 3, 4, and 5-connected maximally planar graphs.

Theorem 4.2. [5] If G is a connected graph of order n and $2 \leq j \leq n$ is an integer, then the following hold.

1. If G is a maximally planar 3-connected graph, then $\text{sdiam}_k(G) \leq \frac{n}{3} + \frac{8n}{3} - 5$.

2. If G is a maximally planar 4-connected graph, then $\text{sdiam}_k(G) \leq \frac{n}{4} + \frac{19n}{4} - 9$.

3. If G is a maximally planar 5-connected graph, then $\text{sdiam}_k(G) \leq \frac{n}{5} + \frac{14n}{5} - 9$.

Furthermore, each of these bounds are asymptotically tight.

We now turn our attention to the n-dimensional hypercube, Q_n, which we refer to as the n-cube. We identify each vertex with a binary string of length n. Here, two vertices are adjacent if their corresponding binary strings differ in exactly one position. We refer to the all zero vertex by $\mathbf{0}$.

In regards to the n-cube, Tao Jiang, Zevi Miller, and Dan Pritikin [37] studied how large the Steiner distance of k vertices can be in the n-cube as $n \to \infty$, while Zevi Miller and Dan Pritikin [44] gave near tight bounds for the Steiner distance of a layer, i.e. vertices with the same number of 1's, in the n-dimensional hypercube Q_n as $n \to \infty$. In general, finding Steiner trees in the n-cube is computationally difficult. In fact, Foulds and Graham showed in 1982 that the problem is NP-hard [24].

In this chapter we give natural upper and lower bounds for the Steiner distance of a *large* vertex set in the hypercube. It turns out that even the second order term in this estimate is close to tight. With these bounds, we determine $sdiam_k(Q_n)$ asymptotically for large k.

4.2 Upper Bound

For the upper bound, we utilize connected dominating sets of Q_n. A set $S \subset V(Q_n)$ is a *dominating set* of Q_n if every vertex of Q_n is either an element of S or has a neighbor in S. The minimum size of all dominating sets is called the *domination number* of Q_n and is denoted $\gamma(Q_n)$. The *connected domination number*, denoted by $\gamma_c(Q_n)$, is the minimum size of all connected dominating sets.

If $n = 2^m - 1$ or $n = 2^m$, $\gamma(Q_n)$ can be determined using coding theory (see [30]). In 1988, Kabatyanskii and Panchenko [38] showed $\frac{\gamma(Q_n)}{2^n/n} \sim 1$. Griggs [29] utilizes this result to show that $\frac{\gamma_c(Q_n)}{2^n/n} \sim 1$ in an upcoming paper. We use this last result to develop an upper bound for the Steiner diameter of subsets of $V(Q_n)$. With results in hand, we are prepared to prove the following lemma.

Lemma 4.3. Suppose that $S \subset V(Q_n)$. Then,

$$d(S) \leq |S| + \frac{2^n}{n}(1 + o(1)).$$

Proof. Begin with a minimal connected dominating set of Q_n. Simply connect each of the elements of S to this connected dominating set. The resulting subgraph spans S and contains at most $|S| + \gamma_c(Q_n) - 1$ edges. Using [29], we then have that $d(S) \leq |S| + \frac{2^n}{n}(1 + o(1))$. □

4.3 Lower Bound

For the lower bound, we make use of the identification of each vertex of the n-cube with a binary string of length n. A vertex in the hypercube is even if its corresponding binary string has an even number of 1's. In the same way a vertex is odd if its corresponding binary string has an odd number of 1's. Figure 4.1 illustrates n-cubes with $1 \leq n \leq 3$. Denote the set of even vertices of a given n-cube by E and the set of odd vertices by $\bar{E}$. The sets E and $\bar{E}$ are are illustrated in Figures 4.2 and 4.3, respectively.

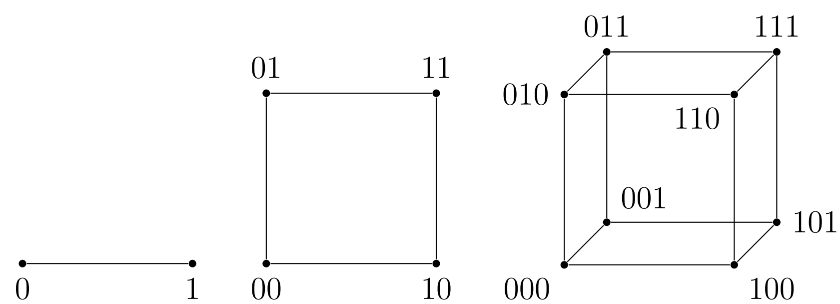

Figure 4.1 The n-cubes for $1 \leq n \leq 3$

Lemma 4.4. Suppose that $S \subseteq V(Q_n)$, be a subset of even vertices i.e. each element of S contains an even number of 1's. Then

$$d(S) \geq |S| + \frac{|S|^2}{n2^n} - 1.$$

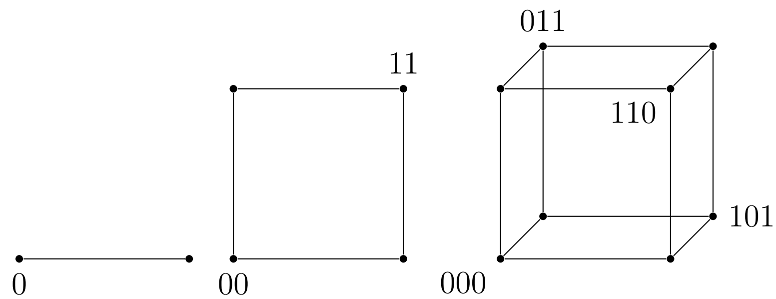

Figure 4.2 The even vertices of the n-cube where $1 \leq n \leq 3$

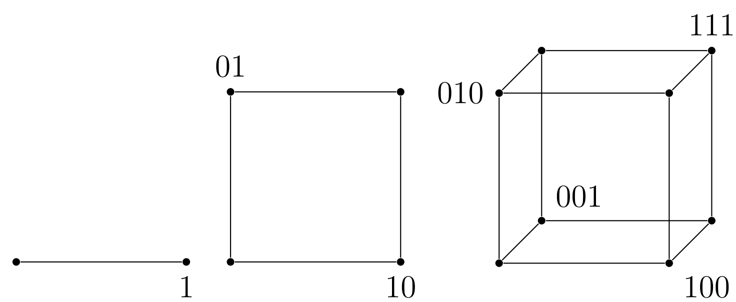

Figure 4.3 The odd vertices of the n-cube where $1 \leq n \leq 3$

Proof. Suppose that $S \subseteq E$, is a subset of the even vertices of Q_n. Let $\bar{S}$ be obtained from S by switching the value of first entry of vertex in S. We have that $d(S) = d(\bar{S})$ by symmetry. Suppose that $T = (V(T), E(T))$ and $\bar{T} = (V(\bar{T}), E(\bar{T}))$ are Steiner trees of S and $\bar{S}$, respectively. Naively, we have that $d(S \cup \bar{S}) \geq 2|S| - 1$. Furthermore, connecting T and $\bar{T}$ yields a subgraph of Q_n spanning $S \cup \bar{S}$. Hence, $d(S \cup \bar{S}) \leq |E(T) \cup E(\bar{T})| + 1$. Then,

$$\begin{aligned} 2|S| - 1 &\leq |E(T) \cup E(\bar{T})| + 1 \\ &= |E(T)| + |E(\bar{T})| - |E(T) \cap E(\bar{T})| + 1 \\ &= 2d(S) - |E(T) \cap E(\bar{T})| + 1, \end{aligned}$$

which implies that

$$2|S| - 2 \leq 2d(S) - |E(T \cap \bar{T})|. \tag{4.1}$$

Let $\Gamma = \langle \alpha, \beta_{i,j} \rangle$ be the subgroup of automorphisms of Q_n generated by the following automorphisms

$$\begin{aligned} \alpha :& v_0 v_1 \cdots v_{n-1} \mapsto v_1 \cdots v_{n-1} v_0 \\ \beta_{i,j} :& v_0 v_1 \cdots v_i \cdots v_j \cdots v_{n-1} \mapsto v_0 v_1 \cdots \bar{v}_i \cdots \bar{v}_j \cdots v_{n-1}. \end{aligned}$$

Here, α cycles through a binary string v while each $\beta_{i,j}$ "flips" i'th and j'th bits of a binary string v.

We now show that for any two edges $e_1, e_2 \in E(Q_n)$, there exists a *unique* $\gamma \in \Gamma$ such that $\gamma(e_2) = e_1$. Suppose that $e_1 = ab$ and $e_2 = uv$ where a and u are even vertices while b and v are odd vertices. Without loss of generality, we may assume that $a = \mathbf{0}$, the vertex of all zeros. This implies that the string b contains a single 1. We shall first prove existence.

First, using a composition of automorphisms of the form $b_{i,j}$ we may map uv to $\mathbf{0}\hat{v}$. Using some power of the automorphism α, we may then map the edge $\mathbf{0}\hat{v}$ to the edge $\mathbf{0}b = e_1$. Hence, there exists $\gamma \in \Gamma$ such that $\gamma(e_2) = e_1$.

To show that this map is unique, we provide a combinatorial argument. Enumerate the images of a binary string $v = v_0 v_1 \cdots v_{n-1}$ under automorphisms in Γ. If $\gamma \in \Gamma$, then

$$\gamma(v) = u_i u_{i+1} \cdots u_n u_1 \cdots u_{i-1},$$

where $u_i \in \{v_i, \bar{v}_i\}$ for each $1 \leq i \leq n$. Now there are n choices for i. Furthermore, $\gamma(v)$ contains an even number of "flipped" bits. So there are 2^{n-1} choices for the number of 1's. So there are $n2^{n-1}$ possible images of v under a $\gamma \in \Gamma$. Since there are $n2^{n-1}$ edges in the n-cube, this implies that $|\gamma| \geq n2^{n-1}$. Hence, any automorphism $\gamma \in \Gamma$ mapping e_1 to e_2 must be unique.

We now consider the experiment of selecting elements $\lambda_1, \lambda_2 \in \Gamma$ with uniform probability and independently applying them to T and $\bar{T}$, respectively. Since for any two edges $f, g \in E(Q_n)$ there there is a unique automorphism mapping an f to g, we have that for any two automorphisms $\lambda_1, \lambda_2 \in \Gamma$, it holds that

$$P[f \in \lambda_1(T)] = P[f \in \lambda_2(T)] = \frac{|E(T)|}{n2^{n-1}}.$$

We now consider the random variable $X = |E(\lambda_1(T) \cap \lambda_2(\bar{T}))|$. For the expected value of X, $\mathbb{E}(X)$, we have that

$$\max_{\lambda_1,\lambda_2}\{|E(\lambda_1(T) \cap \lambda_2(\bar{T}))|\} \geq \mathbb{E}(X).$$

We observe that

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{f \in E(Q_n)} P[f \in \lambda_1(T) \wedge f \in \lambda_2(\bar{T})] \\
&= \sum_{f \in E(Q_n)} \frac{|E(T)|}{n2^{n-1}} \cdot \frac{|E(\bar{T})|}{n2^{n-1}} \\
&= \frac{|E(T)|^2}{n2^{n-1}} \\
&= \frac{d(S)^2}{n2^{n-1}},
\end{aligned}$$

which implies $\max_{\lambda_1,\lambda_2}\{e(\lambda_1(T) \cap \lambda_2(\bar{T}))\} \geq \frac{d(S)^2}{n2^{n-1}}$. Applying this to equation (4.1), we see that

$$2d(S) - \frac{d(S)^2}{n2^{n-1}} \geq 2|S| - 2.$$

Now, $d(S) = |S| + x$ for some $x \geq 0$. So,

$$\begin{aligned}
2(|S| + x) - \frac{(|S| + x)^2}{n2^{n-1}} &\geq 2|S| - 2 \\
2|S| + 2x - \frac{|S|^2 + 2|S|x + x^2}{n2^{n-1}} &\geq 2|S| - 2 \\
2x - \frac{2|S|x}{n2^{n-1}} + 2|S| - \frac{|S|^2 + x^2}{n2^{n-1}} &\geq 2|S| - 2 \\
2x\left(1 - \frac{|S|}{n2^{n-1}}\right) &\geq \frac{|S|^2 + x^2}{n2^{n-1}} - 2 \\
x &\geq \frac{|S|^2}{n2^n} - 1.
\end{aligned}$$

Hence, $d(S) \geq |S| + \frac{|S|^2}{n2^n} - 1$. $\square$

With these preliminary results in hand, we can determine the asymptotic growth of $\mathrm{sdiam}_k(Q_n)$ for large k.

Theorem 4.5. If $k = k(n)$, then

1. If $k = \Omega(2^n)$, then $\mathrm{sdiam}_k(Q_n) = k + O(2^n/n)$, and
2. If $\lim\limits_{n\to\infty} \frac{k}{2^n/n} = \infty$, then $\mathrm{sdiam}_k(Q_n) \sim k$.

Proof. If $k \leq 2^{n-1}$, let $S \subset V(Q_n)$ be a subset of the even vertices of size k. If $k > 2^{n-1}$, let S contain all even vertices and choose the remaining odd vertices randomly. Applying the bounds determined in Lemmas 4.3 and 4.4, we see that

$$k + \frac{k^2}{n2^n} - 1 \leq d(S) \leq \mathrm{sdiam}_k(Q_n) \leq k + \frac{2^n}{n}(1 + o(1)).$$

This immediately gives that $\mathrm{sdiam}_k(Q_n) \geq k - 1 + O(2^n/n)$ and $\mathrm{sdiam}_k(Q_n) = \Omega(k)$. If $\lim\limits_{n\to\infty} \frac{k}{2^n/n} = \infty$, then we have that $2^n/n = o(k)$, so $\mathrm{sdiam}(Q_n) = k(1 + o(1))$, giving $\mathrm{sdiam}_k(Q_n) \sim k$. □

4.4 Directions for future research

The bounds produced in this section agree only in the first two terms. Future work could be applied to two fronts: First, one could attempt to improve these bounds to agree in more than two terms. Second, one could also develop strategies to create bounds for smaller values of k. Given the computational difficulty of finding Steiner trees in the n-cube, it is possible that exact bounds will not be found. However, there exist many graph classes for which bounds on the Steiner k-diameter do not exist for an arbitrary k. A future area of study is to develop bounds for an arbitrary k in other graph classes. Still more, reducing to specific values of k, one may be able to produce tighter bounds for both the n-cube and other graph classes.

Chapter 5

A Midrange Crossing Constant for Certain Graph Classes

5.1 Introduction

Given an undirected graph $G = (V, E)$, a *drawing* of G is a representation of G in the plane such that every edge $uv \in E$ is represented by a simple continuous curve between the points corresponding to u and v, which does not pass through any point representing a vertex of G. For simplicity, we assume that no two curves share infinitely many points, no two curves are tangent to each other, and no three curves pass through the same point. The *crossing number* $\mathrm{cr}(G)$ is defined as the minimum number of crossing points in a drawing of G. It is well-known [55] that in any drawing that realizes the crossing number, any pair of edges crosses in at most one point, and a pair of crossing edges do have 4 distinct endvertices. Computing $\mathrm{cr}(G)$ is an NP-hard problem [25]. Motivated by applications to VLSI design, Leighton [41], and independently Ajtai et al. [2], gave the following general lower bound on the crossing number of a graph, which is better known as the *crossing lemma*.

Theorem 5.1 ([41, 2])**.** For any graph G with n vertices and $e > 4n$ edges, we have

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{e^3}{n^2}.$$

Further improvements on the constant are made by Pach and Tóth [49] and Pach et al [50]. The current best bound is due to Ackerman [1], who showed that $\mathrm{cr}(G) \geq \frac{e^3}{29n^2}$

when $e > 7n$. Székely [56] used the crossing lemma to give a simple proof of the Szemerédi–Trotter theorem on the number of point-line incidences [57].

For a positive integer n and real number $e \geq 0$, let $\kappa(n, e)$ be the minimum crossing number taken over all graphs with n vertices and at least e edges. The crossing lemma implies that for $e > 4n$, $\kappa(n, e)\frac{n^2}{e^3}$ is bounded below by a positive constant. Pach, Spencer and Tóth [48] showed that for $n \ll e \ll n^2$ (i.e. in the midrange),

$$\lim_{n\to\infty} \kappa(n, e)\frac{n^2}{e^3} = C > 0,$$

which proves a conjecture of Erdős and Guy [22], made a decade before the crossing lemma. Here, we use the notation $f(n) \ll g(n)$ to mean that $\lim_{n\to\infty} f(n)/g(n) = 0$. The constant C above is also called the *midrange crossing constant*.

For any positive integer $k \geq 1$, the *k-planar crossing number* $\mathrm{cr}_k(G)$ of G is defined as the minimum of $\sum_{i=1}^{k} \mathrm{cr}(G_i)$, where the minimum is taken over all graphs $G_1, G_2, \ldots, G_k$ such that $\bigcup_{i=1}^{k} E(G_i) = E(G)$. Pach et al. [51] showed a general bound on the ratio of the k-planar crossing number to the (ordinary) crossing number of a graph. They defined

$$\alpha_k = \sup \frac{\mathrm{cr}_k(G)}{\mathrm{cr}(G)},$$

where the supremum is taken over all nonplanar graphs G. Pach et al. [51] showed that for every positive integer k,

$$\frac{1}{k^2} \leq \alpha_k \leq \frac{2}{k^2} - \frac{1}{k^3}.$$

For $k = 2$, this gives the same upper bound of $3/8$ as in [14]. Very recently, Asplund et al. [7] closed the gap and showed that

$$\alpha_k = \frac{1}{k^2}(1 + o(1)) \text{ as } k \to \infty.$$

The lower bound that $\alpha_k \geq \frac{1}{k^2}$ in Pach et al. [51] depends on the existence of the midrange crossing number $C > 0$, though not on its value.

For the family of bipartite graphs, define analogously the constant $\beta_k = \sup \frac{\mathrm{cr}_k(G)}{\mathrm{cr}(G)}$, where the supremum is taken over all *non-planar bipartite* graphs G. Asplund et al. [7] showed that for all positive integers k,

$$\beta_k = \frac{1}{k^2}.$$

As before, the lower bound $\beta \geq \frac{1}{k^2}$, depends on the existence of the midrange crossing constant $C_B > 0$ for the family of bipartite graphs. This motivated us to extend the proof of Pach, Spencer and Tóth [48] to show the existence of the midrange crossing constant $C_{\mathcal{B}}$ for certain graph classes $\mathcal{B}$, which may or not be equal to the midrange crossing constant C for all graphs. As such, the proofs in this chapter closely follow those of the original paper. The current best known bounds for C are $0.034 \leq C \leq 0.09$; see [49, 50, 1], while Angelini, Bekos, Kaufmann, Pfister and Ueckerdt [6] showed that the midrange crossing constant for the class of bipartite graphs is at least $16/289 > 0.055$.

For a class of graphs $\mathcal{B}$, define $\kappa_{\mathcal{B}}(n, e)$ to be the minimum crossing number of a graph in $\mathcal{B}$ with n vertices and at least e edges. The following natural questions arise:

Question 5.2. For a given class $\mathcal{B}$, does there exist a constant $C_{\mathcal{B}}$ such that $\lim_{n\to\infty} \kappa_{\mathcal{B}}(n, e)\frac{n^2}{e^3} = C_{\mathcal{B}}$ in the midrange?

Question 5.3. Are there two classes of graphs $\mathcal{B}$ and $\mathcal{D}$ such that $C_{\mathcal{B}}$ and $C_{\mathcal{D}}$ exist with $C_{\mathcal{B}} \neq C_{\mathcal{D}}$?

Towards answering these questions, we define the following class of graphs.

Definition 5.4. A family of graphs $\mathcal{B}$ is a PST-class (abbreviating Pach, Spencer and Tóth) if it satisfies the following properties:

1. $\mathcal{B}$ contains a graph with at least one edge;

2. $\mathcal{B}$ is closed under taking subgraphs;

3. $\mathcal{B}$ is closed under taking the union of disjoint copies of graphs in $\mathcal{B}$;

4. $\mathcal{B}$ is closed under vertex cloning, i.e. $\mathcal{B}$ is closed under replacing a vertex v with vertices $v_1, \dots, v_m$ such that $N(v_i) = N(v)$ for all $1 \leq i \leq m$.

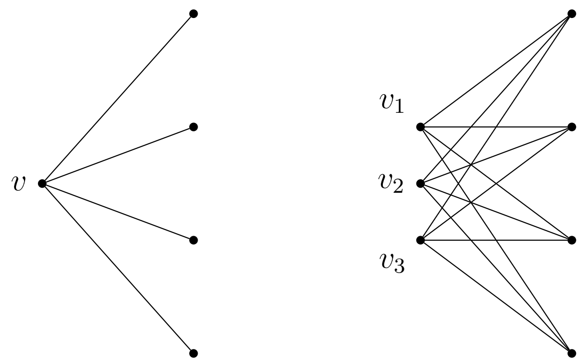

Figure 5.1 The vertex v is "cloned" into the vertices v_1, v_2, and v_3.

Note that properties (2) and (4) imply that a PST-class $\mathcal{B}$ is also closed under vertex-splitting i.e. $\mathcal{B}$ is closed under replacing a vertex v with vertices $v_1, v_2, \dots, v_m$ such that $N(v) = \bigcup_{i=1}^{m} N(v_i)$ and $N(v_i) \cap N(v_j) = \emptyset$ for $1 \leq i < j \leq m$. Figures 5.1 and 5.2 illustrate the cloning and splitting of a vertex, respectively.

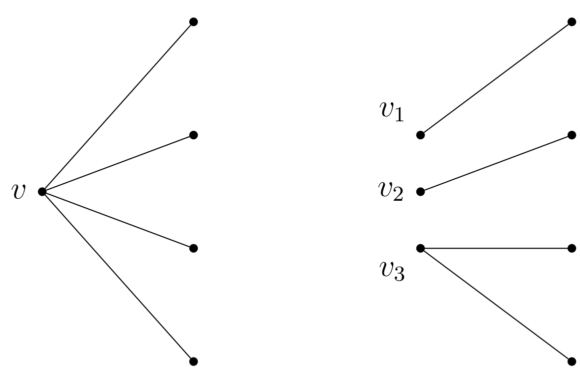

Figure 5.2 The vertex v is "split" into the vertices v_1, v_2, and v_3.

PST classes form a lattice with respect to the subclass relation, with the set of bipartite graphs being the minimum and the set of all graphs being a maximum element of the class:

Theorem 5.5. Any PST-class contains the family of bipartite graphs as a subclass, and the intersection of two PST-classes is also a PST class. Moreover, the following are PST classes:

1. ℓ-colorable graphs;

2. K_t-free graphs ($t \geq 3$);

3. graphs without odd cycles shorter than g.

Careful examination of the proof presented in [48] yields that PST-classes have midrange crossing constants. In this chapter, we will prove the following result.

Theorem 5.6. If $\mathcal{B}$ is a PST-class, then $C_{\mathcal{B}}$ exists, i.e. there is a constant $C_{\mathcal{B}} > 0$ such that in the midrange

$$\lim_{n\to\infty} \kappa_{\mathcal{B}}(n,e)\frac{n^2}{e^3} = C_{\mathcal{B}}.$$

For Question 5.3, the answer remains elusive. By Theorem 5.5, if we restrict our attention to PST-classes, an affirmative answer implies that the midrange crossing constant for the class of bipartite graphs is bigger than the midrange crossing constant for all graphs, which we tend to believe.

Pach et al. [51] pointed out that the arguments in [48] can be repeated for rectilinear drawings of graphs and rectilinear crossing numbers; therefore a midrange rectilinear crossing constant exists, which is not necessarily the same as the midrange crossing constant. Extending this argument, Theorems 5.6 and 5.5 have their rectilinear versions, with possibly different constants.

The proof of Theorem 5.6 closely follows the original proof for the existence of the midrange crossing constant presented in [48], with emphasis added on the required properties of PST classes of graphs.

5.2 Proof of Theorem 5.5

If $\mathcal{B}$ is a PST-class, properties (1) and (2) imply that $K_2 \in \mathcal{B}$. Property (4) then gives that $K_{n,m} \in \mathcal{B}$ for all n, m, and property (2) then implies that all bipartite graphs are elements of $\mathcal{B}$. Let $\mathcal{B}_1, \mathcal{B}_2$ be PST-classes, then their intersection must contain all bipartite graphs, and therefore a graph, which is not edgeless. If G^* is a graph obtained from a graph in $G \in \mathcal{B}_1 \cap \mathcal{B}_2$ by taking subgraphs, or vertex cloning, then, as $\mathcal{B}_1$ and $\mathcal{B}_2$ are closed under these operations, $G^* \in \mathcal{B}_1 \cap \mathcal{B}_2$. As $\mathcal{B}_1, \mathcal{B}_2$ are closed under taking disjoint unions, so is $\mathcal{B}_1 \cap \mathcal{B}_2$. Hence, $\mathcal{B}_1 \cap \mathcal{B}_2$ is a PST-class.

K_2 is in the classes of ℓ-colorable graphs, K_t-free graphs for $t \geq 3$, and graphs without odd cycles shorter then g, so these three classes satisfy property (1). It is obvious that taking a subgraph does not increase the chromatic number, the clique number or the length of the shortest odd cycle. The chromatic number, clique number and length of the shortest odd cycle are the minimum of these quantities respectively over the components of a graph, so these classes are closed under property (3).

Let G^* be obtained from G by cloning v into $v_1, \ldots, v_m$. Given a good ℓ-coloring of G, we can obtain a good ℓ-color of G^* by assigning the color of v to all v_i; any complete subgraph of G^* can contain at most one v_i and thus correspond to a complete subgraph of the same order in G. Any cycle in G^* that contains at most one of the v_i's corresponds to a cycle of the same length in G. Any cycle in G^* that contains more than one v_i corresponds to a closed walk in G that visits all vertices (of the walk) but v at most once, so it corresponds to the union of cycles in G; this means that for any odd cycle in G^* we can find an odd cycle in G that is not longer. This shows that these classes are all PST-classes.

5.3 Proof of Theorem 5.6

Throughout this section, $\mathcal{B}$ denotes a PST-class. We first prove a lemma that bounds the minimum number of crossings of a graph in $\mathcal{B}$ with n vertices and linearly many edges.

Lemma 5.7. For any $a \geq 4$, and $n \geq 2a+1$,

$$\frac{a^3}{100} \leq \frac{\kappa_{\mathcal{B}}(n, an)}{n} \leq 8a^3.$$

Proof. It is known (see [49]) that for any $a \geq 4$, $\frac{a^3 n}{100} \leq \kappa(n, an) \leq a^3 n$. Since $\kappa(n, an) \leq \kappa_{\mathcal{B}}(n, an)$ for any $a > 0$, it follows that $\frac{a^3 n}{100} \leq \kappa_{\mathcal{B}}(n, an)$.

Since $\kappa(n, 2an) \leq 8a^3 n$, we know that there exists a graph G on n vertices and $\lceil 2an \rceil$ edges such that $\mathrm{cr}(G) \leq 8a^3 n$. As it is well known, every graph has a bipartite subgraph with at least half as many edges. Hence there exists a bipartite subgraph H of G, such that H has at least an edges and $\mathrm{cr}(H) \leq \mathrm{cr}(G) \leq 8a^3 n$. Since $\mathcal{B}$ is closed under taking subgraphs, $H \in \mathcal{B}$. Thus, for any $a \geq 4$, we have that

$$\frac{a^3}{100} \leq \frac{\kappa_{\mathcal{B}}(n, an)}{n} \leq \frac{\mathrm{cr}(H)}{n} \leq 8a^3.$$

$\square$

With this lemma in hand, we now follow the proof presented in [48] restricting ourselves to the graph class PST. To begin, we prove the following lemma.

Lemma 5.8. We have the following:

(a) For any $a > 0$, the limit

$$\gamma_{\mathcal{B}}[a] = \lim_{n \to \infty} \frac{\kappa_{\mathcal{B}}(n, na)}{n}$$

exists and is finite.

(b) $\gamma_{\mathcal{B}}[a]$ is a convex function.

(c) For any $a \geq 4$, $1 > \delta > 0$,

$$\gamma_{\mathcal{B}}[a] - \gamma_{\mathcal{B}}[a(1-\delta)] \leq \gamma_{\mathcal{B}}[a(1+\delta)] - \gamma_{\mathcal{B}}[a] \leq 10^4 \delta \gamma_{\mathcal{B}}[a].$$

(d) $\gamma_{\mathcal{B}}[a]$ is continuous.

Proof. Let $G_1, G_2 \in \mathcal{B}$, and let G_3 be their disjoint union. As $\mathrm{cr}(G_3) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$ and $\mathcal{B}$ is closed under taking disjoint union,

$$\kappa_{\mathcal{B}}(n_1 + n_2, e_1 + e_2) \leq \kappa_{\mathcal{B}}(n_1, e_1) + \kappa_{\mathcal{B}}(n_2, e_2).$$

This implies that $f_a(n) := \kappa_{\mathcal{B}}(n, na)$ is subadditive. Hence, by Fekete's Lemma on subadditive sequences (see [53] Ex. 98), the limit

$$\gamma_{\mathcal{B}}[a] = \lim_{n \to \infty} \frac{\kappa_{\mathcal{B}}(n, na)}{n}$$

exists and is finite for every $a > 0$. This concludes the proof of part (a).

For part (b), let $0 < \alpha < 1$. Suppose $a, b > 0$ with n such that αn is also an integer. Then,

$$\begin{aligned}
\kappa_{\mathcal{B}}(n, (\alpha a + (1-\alpha) b) n) &= \kappa_{\mathcal{B}}(\alpha n + (1-\alpha) n, (\alpha a + (1-\alpha) b) n) \\
&\leq \kappa_{\mathcal{B}}(\alpha n, \alpha a n) + \kappa_{\mathcal{B}}((1-\alpha) n, (1-\alpha) b n).
\end{aligned}$$

This implies that

$$\begin{aligned}
\gamma_{\mathcal{B}}[\alpha a + (1-\alpha) b] &= \lim_{n \to \infty} \frac{\kappa_{\mathcal{B}}(n, (\alpha a + (1-\alpha) b) n)}{n} \\
&\leq \lim_{n \to \infty, \alpha n \in \mathbb{Z}} \frac{\kappa_{\mathcal{B}}(\alpha n, \alpha a n)}{n} + \lim_{n \to \infty, \alpha n \in \mathbb{Z}} \frac{\kappa_{\mathcal{B}}((1-\alpha) n, (1-\alpha) b n)}{n}.
\end{aligned}$$

Now, for any rational number α, let $m = \alpha n$ and $q = (1-\alpha) n = n - m$. Then,

$$\begin{aligned}
&\lim_{n \to \infty, \alpha n \in \mathbb{Z}} \frac{\kappa_{\mathcal{B}}(\alpha n, \alpha a n)}{n} + \lim_{n \to \infty, \alpha n \in \mathbb{Z}} \frac{\kappa_{\mathcal{B}}((1-\alpha) n, (1-\alpha) b n)}{n} \\
&= \lim_{m \to \infty} \frac{\kappa_{\mathcal{B}}(m, am)}{(m/\alpha)} + \lim_{q \to \infty} \frac{\kappa_{\mathcal{B}}(q, bq)}{q/(1-\alpha)} \\
&= \alpha \lim_{m \to \infty} \frac{\kappa_{\mathcal{B}}(m, am)}{m} + (1-\alpha) \lim_{q \to \infty} \frac{\kappa_{\mathcal{B}}(q, bq)}{q} \\
&= \alpha \gamma_{\mathcal{B}}[a] + (1-\alpha) \gamma_{\mathcal{B}}[b].
\end{aligned}$$

Hence, for any rational α

$$\gamma_{\mathcal{B}}[\alpha a + (1-\alpha)b] \le \alpha\gamma_{\mathcal{B}}[a] + (1-\alpha)\gamma_{\mathcal{B}}[b].$$

We now relax the restriction that α is rational. Suppose that $a \le b$ and let $\{\alpha_n\}_{n=0}^{\infty}$ be a monotone increasing sequence of rational numbers converging to α. For any $n \ge 0$,

$$\begin{aligned}\gamma_{\mathcal{B}}[\alpha a + (1-\alpha)b] &\le \gamma_{\mathcal{B}}[\alpha_n a + (1-\alpha_n)b] \\ &\le \alpha_n\gamma_{\mathcal{B}}[a] + (1-\alpha_n)\gamma_{\mathcal{B}}[b],\end{aligned}$$

which converges to $\alpha\gamma_{\mathcal{B}}[a] + (1-\alpha)\gamma_{\mathcal{B}}[b]$. So,

$$\gamma_{\mathcal{B}}[\alpha a + (1-\alpha)b] \le \alpha\gamma_{\mathcal{B}}[a] + (1-\alpha)\gamma_{\mathcal{B}}[b]$$

for all real numbers $0 < \alpha < 1$. That is, $\gamma_{\mathcal{B}}$ is convex, so part (b) is true. By convexity, we then have

$$\gamma_{\mathcal{B}}[a] - \gamma_{\mathcal{B}}[a(1-\delta)] \le \gamma_{\mathcal{B}}[a(1+\delta)] - \gamma_{\mathcal{B}}[a],$$

which is the first inequality in part (c). For the second inequality, by Lemma 5.7, we obtain the inequality

$$\frac{a^3}{100} \le \gamma_{\mathcal{B}}(a) \le 8a^3.$$

Then, for $a \ge 4$ and $0 < \delta < 1$,

$$\begin{aligned}\gamma_{\mathcal{B}}[(1+\delta)a] &= \gamma_{\mathcal{B}}[(1-\delta)a + 2\delta a] \\ &\le (1-\delta)\gamma_{\mathcal{B}}[a] + \delta\gamma_{\mathcal{B}}[2a] \\ &\le \gamma_{\mathcal{B}}[a] + \delta\gamma_{\mathcal{B}}[2a].\end{aligned}$$

Hence,

$$\gamma_{\mathcal{B}}[(1+\delta)a] - \gamma_{\mathcal{B}}[a] \le \delta\gamma_{\mathcal{B}}[2a] \le \delta \cdot 64a^3 \le \delta \cdot 6400\gamma_{\mathcal{B}}[a] < 10^4\delta \cdot \gamma_{\mathcal{B}}[a],$$

which finishes the proof of part (c) and implies part (d). $\square$

We now set

$$D_{\mathcal{B}} := \limsup_{a\to\infty} \frac{\gamma_{\mathcal{B}}[a]}{a^3}.$$

Note that by Lemma 5.7 we have $0.01 \le D_{\mathcal{B}} \le 8$. We will follow the proof of Theorem 5.6 presented in [48] by showing two lemmas that imply that $D_{\mathcal{B}}$ is the midrange crossing constant $C_{\mathcal{B}}$.

Lemma 5.9. For any $0 < \epsilon < 0.1$, there exists $N = N(\epsilon)$ such that $\kappa_{\mathcal{B}}(n,e) > (1-\epsilon)\frac{e^3}{n^2}D_{\mathcal{B}}$, whenever $\min\{n, e/n, n^2/e\} > N$.

Proof. Let $A > \frac{2\cdot 10^9}{\epsilon^3}$ be a rational number satisfying

$$\frac{\gamma_{\mathcal{B}}[A]}{A^3} > D_{\mathcal{B}}\left(1-\frac{\epsilon}{10}\right).$$

Such a number exists by the definition of $D_{\mathcal{B}}$. Let $N = N(\epsilon) \ge A$ such that if $n > N$, $e = nA'$, and $|A - A'| \le A\epsilon$, then

$$\kappa_{\mathcal{B}}(n,e) > \gamma_{\mathcal{B}}[A']\left(1-\frac{\epsilon}{10}\right)n. \tag{5.1}$$

Such an N certainly exists, as this is equivalent to

$$\frac{\kappa_{\mathcal{B}}(n,e)}{n} > \lim_{k\to\infty}\frac{\kappa_{\mathcal{B}}(k,A'k)}{k}\left(1-\frac{\epsilon}{10}\right),$$

for all $n \ge N$.

Let n and e be fixed integers so that $\min\{n, e/n, n^2/e\} > N$ and let $G = (V,E)$ be a graph with $|V| = n$ vertices and $|E| = e$ edges, which can be drawn in the plane with $\kappa_{\mathcal{B}}(n,e)$ crossings. Let U be a random independently chosen subset of V with $P[u \in U] = p$, where

$$p = \frac{An}{e} > \frac{2\cdot 10^9 n}{e\cdot\epsilon^3}.$$

Let $\nu = |U|$ and let η and ξ be the number of edges and crossings (in the drawing) of the graph induced by U. We have that ν has expected value np and variance $p(1-p)n \le pn$. By Chebyshev's inequality, we have that

$$\Pr\left[|\nu - pn| > \frac{\epsilon}{10^4}pn\right] \le \frac{10^8 pn(1-p)}{(pn)^2\epsilon^2} \le \frac{10^8}{pn\epsilon^2} \le \frac{\epsilon^3 10^8 e}{2\cdot 10^9 n^2\epsilon^2} < \frac{\epsilon}{10}.$$

We note that $\eta = \sum_{u,v \in G} I_{uv}$ where I_{uv} is the indicator function for the event $u, v \in U$. Then, $E[\eta] = ep^2$. Since I_{uv} is an indicator function, we have that

$$E[I_{uv}] = p^2$$
$$\mathrm{Var}[I_{uv}] = p^2(1-p^2)$$
$$\mathrm{Cov}[I_{uv}, I_{wx}] = \Pr[(u, v \in U) \cap (w, x \in U)] - \Pr(u, v \in U) \cdot \Pr(w, x \in U).$$

It is immediate that $\mathrm{Var}(I_{uv}) \le E(I_{uv})$ and that $\mathrm{Cov}[I_{uv}, I_{wx}] = 0$ if $\{u, v\} \cap \{w, x\} = \emptyset$. Hence,

$$\mathrm{Var}[\eta] = \sum_{uv \in E} \mathrm{Var}[I_{uv}] + \sum_{uv, uw \in E} \mathrm{Cov}[I_{uv}, I_{uw}],$$

and

$$\sum_{uv \in E} \mathrm{Var}[I_{uv}] \le \sum_{uv \in E} E[I_{uv}] = E[\eta] = ep^2$$

Using the bound $\mathrm{Cov}[I_{uv}, I_{uw}] \le E[I_{uv} I_{uw}] = p^3$, we see that

$$\mathrm{Var}[\eta] \le p^2 e + p^3 \sum_{v \in V} \binom{\deg(v)}{2}.$$

Since $\deg(v) < n$ and $\sum_{v \in V} \deg(v) = 2e$, we have that

$$\sum_{v \in V} \binom{\deg(v)}{2} \le \frac{1}{2} \sum_{v \in V} d^2(v) < \frac{1}{2} \sum_{v \in V} \deg(v) n = en.$$

Hence, using that $pn = \frac{An^2}{e} > AN > 1$,

$$\mathrm{Var}[\eta] \le p^2 e + p^3 en \le 2p^3 en.$$

Applying the Chebyshev Inequality, and using that $pe = An > \frac{2 \cdot 10^9}{\epsilon^3}$, we see that

$$\Pr\left[|\eta - p^2 e| > \frac{\epsilon}{10^4} p^2 e\right] \le \frac{2p^3 en}{\frac{\epsilon^2}{10^8} p^4 e^2} = \frac{2n}{\frac{\epsilon^2}{10^8} pe} < \frac{2}{\frac{\epsilon^2}{10^8} \frac{2 \cdot 10^9}{\epsilon^3}} = \frac{\epsilon}{10}.$$

This implies that with probability at least $1 - \frac{\epsilon}{5}$,

$$pn\left(1 - \frac{\epsilon}{10^4}\right) < \nu < pn\left(1 + \frac{\epsilon}{10^4}\right), \text{ and } p^2 e\left(1 - \frac{\epsilon}{10^4}\right) < \eta < p^2 e\left(1 + \frac{\epsilon}{10^4}\right).$$

Hence, with probability at least $1 - \frac{\epsilon}{5}$,

$$\frac{p^2 e}{pn} \cdot \frac{1 - \frac{\epsilon}{10^4}}{1 + \frac{\epsilon}{10^4}} < \frac{\eta}{\nu} < \frac{p^2 e}{pn} \cdot \frac{1 + \frac{\epsilon}{10^4}}{1 - \frac{\epsilon}{10^4}},$$

which implies that

$$A\left(1-\frac{3\epsilon}{10^4}\right) < \frac{\eta}{\nu} < A\left(1+\frac{3\epsilon}{10^4}\right).$$

Now we set $A' = \frac{\eta}{\nu}$. The subgraph induced by U has ν vertices and $A'\nu = \eta$ edges. So, with probability at least $1-\frac{\epsilon}{5}$, equation (5.1) implies that the number of crossings in this induced subgraph is at least

$$\nu\gamma_{\mathcal{B}}[A']\left(1-\frac{\epsilon}{10}\right) \geq pn\left(1-\frac{\epsilon}{10}\right)\gamma_{\mathcal{B}}[A']\left(1-\frac{\epsilon}{10}\right).$$

Then, the expected number of crossings in the subgraph induced by U in G is at least

$$\begin{aligned}
E[\xi] &\geq \left(1-\frac{\epsilon}{5}\right)pn\left(1-\frac{\epsilon}{10}\right)\gamma_{\mathcal{B}}[A']\left(1-\frac{\epsilon}{10}\right)\\
&\geq \left(1-\frac{\epsilon}{5}\right)pn\left(1-\frac{\epsilon}{10}\right)\gamma_{\mathcal{B}}[A]\left(1-\frac{3\epsilon}{10}\right)\left(1-\frac{\epsilon}{10}\right)\\
&> \left(1-\frac{\epsilon}{5}\right)pn\left(1-\frac{\epsilon}{10}\right)D_{\mathcal{B}}A^3\left(1-\frac{3\epsilon}{10}\right)\left(1-\frac{\epsilon}{10}\right)\left(1-\frac{\epsilon}{10}\right)\\
&\geq (1-\epsilon)D_{\mathcal{B}}A^3pn.
\end{aligned}$$

However, since each crossing lies in $G[U]$ with probability p^4, we know that

$$E[\xi] = p^4\kappa_{\mathcal{B}}(n,e).$$

Hence,

$$\kappa_{\mathcal{B}}(n,e) \geq (1-\epsilon)\frac{pnD_{\mathcal{B}}A^3}{p^4} = \frac{e^3}{n^2}(1-\epsilon)D_{\mathcal{B}}.$$

□

Lemma 5.10. For any $0 < \epsilon < 0.1$, there exists $M = M(\epsilon)$ such that $\kappa_{\mathcal{B}}(n,e) < (1+\epsilon)\frac{e^3}{n^2}D_{\mathcal{B}}$, whenever $\min\{n, e/n, n^2/e\} > M$.

Proof. Let $A > 10^8/\epsilon^2$ be a rational number such that $A^{3/2}$ is an integer satisfying

$$D_{\mathcal{B}}\left(1-\frac{\epsilon}{10}\right) < \frac{\gamma_{\mathcal{B}}[A]}{A^3} < D_{\mathcal{B}}\left(1+\frac{\epsilon}{10}\right).$$

Let $M_1 = M_1(\epsilon) \geq A$ such that, if $n > M_1$ and $e = nA$, then

$$D_{\mathcal{B}}A^3n\left(1-\frac{\epsilon}{5}\right) < \kappa_{\mathcal{B}}(n,e) < D_{\mathcal{B}}A^3n\left(1+\frac{\epsilon}{5}\right).$$

Let G_1 be a graph in $\mathcal{B}$ with $n_1 > M_1$ vertices $e_1 = An_1$ edges, and suppose that G_1 is drawn in the plane with $\kappa_{\mathcal{B}}(n_1, e_1)$ crossings, where

$$D_{\mathcal{B}} A^3 n_1 \left(1 - \frac{\epsilon}{5}\right) < \kappa_{\mathcal{B}}(n_1, e_1) < D_{\mathcal{B}} A^3 n_1 \left(1 + \frac{\epsilon}{5}\right).$$

For each vertex $v \in G_1$ do the following. Let $\deg(v) = r_v A^{3/2} + s_v$ where r_v, s_v are integers and $0 \leq s_v < A^{3/2}$. We split v into $r_v + 1$ vertices, one with degree s_v and r_v with degree $A^{3/2}$. (Note that this implies that we do nothing for vertices with $r_v = 0$, i.e. when the degree of the vertex is smaller than $A^{3/2}$.) Drawing these vertices very close to each other, we may do this without creating any additional crossings. This creates a drawing of a new graph G_2 that has n_2 vertices, e_1 edges and maximum degree at most $A^{3/2}$, and the crossing number of this drawing is $\kappa_{\mathcal{B}}(n_1, e_1)$. Since $\mathcal{B}$ is closed under vertex splitting, $G_2 \in \mathcal{B}$. We have that

$$2An_1 = 2e_1 = \sum_{v \in G_1} \deg(v) = \sum_{v \in V} (r_v A^{3/2} + s_v),$$

which implies that $\sum\limits_{v \in V} r_v \leq \frac{2n_1}{\sqrt{A}}$. Hence,

$$n_1 \leq n_2 \leq n_1 + \frac{2n_1}{\sqrt{A}} < n_1 \left(1 + \frac{\epsilon}{10}\right).$$

Now, fix integers n and e such that $\min\{n, e/n, n^2/e\} > M(\epsilon) = 10\frac{M_1}{\epsilon}$. Let

$$L = \frac{e/n}{e_1/n_2} \quad \text{and} \quad K = \frac{n^2/e}{n_2^2/e_1},$$

so that $n = KLn_2$ and $e = KL^2 e_1$. Let

$$\tilde{L} = \left\lfloor L\left(1 + \frac{\epsilon}{10}\right)\right\rfloor \quad \text{and} \quad \tilde{K} = \left\lfloor K\left(1 - \frac{\epsilon}{10}\right)\right\rfloor,$$

and let

$$\tilde{n} = \tilde{K}\tilde{L}n_2 \quad \text{and} \quad \tilde{e} = \tilde{K}\tilde{L}^2 e_1.$$

Then,

$$\tilde{n} < n \text{ and } e < \tilde{e}.$$

which implies that $\kappa_{\mathcal{B}}(n,e) < \kappa_{\mathcal{B}}(\tilde{n},\tilde{e})$. Clone each vertex of G_2 into $\tilde{L}$ very close vertices, and substitute each edge of G_2 with the corresponding $\tilde{L}^2$ edges, each very close to the original, obtaining a drawing of a new graph G_3 with $n_2\tilde{L}$ vertices and $e_1\tilde{L}^2$ edges. As G_3 is obtained from G_2 by cloning each vertex, $G_3 \in \mathcal{B}$. Then make $\tilde{K}$ copies of this drawing, each separated from the others. We then have a graph $\tilde{G}$ on $\tilde{n}$ vertices and $\tilde{e}$ edges drawn in the plane. As $\tilde{G}$ is a disjoint union of $\tilde{K}$ copies of G_3, $\tilde{G} \in \mathcal{B}$. We will estimate the number of crossings X in this drawing of $\tilde{G}$.

A crossing in the original drawing of G_2 corresponds to $\tilde{K}\tilde{L}^4$ crossings in the present drawing of $\tilde{G}$. For any two edges of G_2 with a common endpoint, the edges arising from them have at most $\tilde{K}\tilde{L}^4$ crossings with each other. So,

$$X \le \tilde{K}\tilde{L}^4\left(\kappa_{\mathcal{B}}(n_1,e_1) + \sum_{v\in V(G_2)}\binom{\deg(v)}{2}\right).$$

However,

$$\sum_{v\in V(G_2)} d_{G_2}(v) = 2e_1 = 2An_1$$

and $d_{G_2}(v) \le A^{3/2}$, so

$$\sum_{v\in V(G_2)}\binom{d_{G_2}(v)}{2} < \sum_{v\in V(G_2)}\frac{2An_1\cdot A^{3/2}}{2} \le A^{5/2}n_1 < \frac{\kappa_{\mathcal{B}}(n_1,e_1)}{A^{1/2}D_{\mathcal{B}}\left(1-\frac{\epsilon}{5}\right)} \le \frac{\epsilon}{10}\kappa_{\mathcal{B}}(n_1,e_1).$$

Therefore,

$$\kappa_{\mathcal{B}}(\tilde{n},\tilde{e}) \le X < \tilde{K}\tilde{L}^4\kappa_{\mathcal{B}}(n_1,e_1)\left(1+\frac{\epsilon}{10}\right)$$

Putting all these inequality together, we have

$$\begin{aligned}
\kappa_{\mathcal{B}}(n,e) &< \kappa_{\mathcal{B}}(\tilde{n},\tilde{e}) < \tilde{K}\tilde{L}^4\kappa_{\mathcal{B}}(n_1,e_1)\left(1+\frac{\epsilon}{10}\right) < \tilde{K}\tilde{L}^4D_{\mathcal{B}}A^3n_1\left(1+\frac{\epsilon}{5}\right)\left(1+\frac{\epsilon}{10}\right)\\
&= \tilde{K}\tilde{L}^4D_{\mathcal{B}}\frac{e_1^3}{n_1^2}\left(1+\frac{\epsilon}{5}\right)\left(1+\frac{\epsilon}{10}\right) < KL^4D_{\mathcal{B}}\frac{e_1^3}{n_2^2}\left(1+\frac{\epsilon}{10}\right)^4\left(1+\frac{\epsilon}{5}\right)\left(1+\frac{\epsilon}{10}\right)\\
&= \frac{e^3}{n^2}D_{\mathcal{B}}\left(1+\frac{\epsilon}{10}\right)^5\left(1+\frac{\epsilon}{5}\right) < (1+\epsilon)\frac{e^3}{n^2}D_{\mathcal{B}}.
\end{aligned}$$

□

Putting Lemmas 5.9 and 5.10 together and letting ϵ run to 0, we have have proven Theorem 5.6.

5.4 Directions for future research

With Theorem 5.6 in hand, we have a candidate with which to explore question 5.3. Since we do not yet know the true value of the midrange crossing constant of all graphs, an immediate strategy is to determine a lower bound for the midrange crossing constant of a PST class of graphs and compare with the standard midrange crossing constant. Since the class of bipartite graphs is the minimal element in the lattice of PST classes of graphs, the largest midrange crossing constant must be for the class of bipartite graphs. In the same way, the class of all graphs is maximal in the same lattice and must contain the minimal midrange crossing constant. If two PST classes of graphs have different midrange crossing constants, these classes are the obvious candidates.

Beyond this, a further question is to determine which conditions of those listed in Definition 5.4 are necessary to ensure that a class of graphs has a midrange crossing constant. Our proof uses all conditions that determine a PST class. A further question is whether a new proof can be found that eliminates some of these conditions, finding more graph classes that have midrange crossing constants.

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